Math 562 Fall 2020

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Outline

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HW1 is posted on my homepage. I also setup HW1 in the course Moodle page. You need to submit your HW1 via Moodle. The due date for HW1 is 09/08 at noon.

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Theorem 2.9 (Kolmogorov's continuity theorem)

Let $X = (X_t)_{t\in I}$ be a process indexed by a bounded interval of $\mathbb{R}$ and taking values in a complete metric space (E, d). Assume that there exist positive reals q, ϵ, C such that, for every $s, t \in I$,

$$\mathbb{E}[d(X_s, X_t)^q] \leq C|t-s|^{1+\epsilon}.$$

Then, there is a modification $\widetilde{X}$ of X whose sample paths are Hölder continuous with exponent α for every $\alpha \in (0, \frac{\epsilon}{q})$. This means that, for every $\omega \in \Omega$ and $\alpha \in (0, \frac{\epsilon}{q})$, there exists a finite constant $C_\alpha(\omega) > 0$ such that, for every $s, t \in I$,

$$d(\widetilde{X}_s(\omega), \widetilde{X}_t(\omega)) \leq C_\alpha(\omega)|t-s|^\alpha.$$

In particular, $\widetilde{X}$ is a modification of X with continuous sample paths. Such a modification is unique up to indistinguishability.

Remarks

(i) If I is unbounded, we get from this theorem that X has a modification whose sample paths are locally Hölder with exponent α for every $\alpha \in (0, \frac{\epsilon}{q})$.
(ii) It suffices to prove that, for every fixed $\alpha \in (0, \frac{\epsilon}{q})$, X has a modification whose sample paths are Hölder with exponent α. Indeed, we can then apply this result to every choice of α in a sequence $\alpha_k \uparrow \frac{\epsilon}{q}$, noting that the resulting modifications are indistinguishable.

Before we prove Theorem 2.9, we prove a lemma first. Let

$$D = \{j \cdot 2^{-n} : j = 0, 1, \ldots, 2^n - 1, n \geq 1\}.$$

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Lemma 2.10

Let f be a map from D to a metric space (E, d). Assume that there exist $\alpha > 0$ and $K > 0$ such that for every integer $n \geq 1$ and every $j \in \{1, 2, \ldots, 2^n - 1\}$,

$$d(f((j-1) \cdot 2^{-n}), f(j \cdot 2^{-n})) \leq K2^{-n\alpha}.$$

Then, for every $s, t \in D$,

$$d(f(s), f(t)) \leq \frac{K}{1 - 2^{-\alpha}} |t - s|^{\alpha}.$$

Proof of Lemma 2.10

Fix $s, t \in D$ with $s < t$. Let $p \geq 1$ be the smallest integer such that $2^{-p} \leq t - s$, and let $k \geq 0$ be the smallest integer such that $k \cdot 2^{-p} \geq s$. Then, we may write

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Proof of Lemma 2.10 (cont)

$$
\begin{aligned}
s =& k \cdot 2^{-p} - \epsilon_1 \cdot 2^{-p-1} - \cdot - \epsilon_l \cdot 2^{-p-l}, \\
t =& k \cdot 2^{-p} + \epsilon_1' \cdot 2^{-p-1} + \cdot + \epsilon_m' \cdot 2^{-p-m}
\end{aligned}
$$

where l, m are non-negative integers and and $\epsilon_i, \epsilon_j' = 0$ or 1 for every $1 \leq i \leq l$ and $0 \leq j \leq m$. Define, for $1 \leq i \leq l$ and $0 \leq j \leq m$,

$$
\begin{aligned}
s_i =& k \cdot 2^{-p} - \epsilon_1 \cdot 2^{-p-1} - \cdot - \epsilon_i \cdot 2^{-p-i}, \\
t_j =& k \cdot 2^{-p} + \epsilon_1' \cdot 2^{-p-1} + \cdot + \epsilon_j' \cdot 2^{-p-j}
\end{aligned}
$$

Then, noting that $s = s_l$ and $t = t_m$, and that we can apply the assumption of the lemma to each of the pairs (s_0, t_0), (s_{i-1}, s_i) (for $1 \leq i \leq l$) and (t_{j-1}, t_j) (for $1 \leq j \leq m$), we get

Proof of Lemma 2.10 (cont)

$$
\begin{aligned}
d(f(s), f(t)) &= d(f(s_l), f(t_m)) \\
&\leq d(f(s_0), f(t_0)) + \sum_{i=1}^{l} d(f(s_{i-1}), f(s_i)) + \sum_{j=1}^{m} d(f(t_{j-1}), f(t_j)) \\
&\leq K \cdot 2^{-p\alpha} + \sum_{i=1}^{l} K \cdot 2^{-(p+i)\alpha} + \sum_{j=1}^{m} K \cdot 2^{-(p+j)\alpha} \\
&\leq 2K(1 - 2^{-\alpha})^{-1} 2^{-p\alpha} \\
&\leq 2K(1 - 2^{-\alpha})^{-1} (t - s)^{\alpha},
\end{aligned}
$$

since $2^{-p} \leq t - s$. The proof is complete.

Proof of Theorem 2.9

Without loss of generality, we take $I = [0,1]$. Fix $\alpha \in (0, \frac{\epsilon}{q})$. By assumption, for any $a > 0$ and $s, t \in I$,

$$\mathbb{P}(d(X_s, X_t) > a) \leq a^{-q}\mathbb{E}[d(X_s, X_t)^q] \leq Ca^{-q}|t-s|^{1+\epsilon}.$$

Applying this to $s = (i-1)\cdot 2^{-n}, t = i \cdot 2^{-n}$ (for $i = 1, 2, \ldots, 2^n$) and $a = 2^{-n\alpha}$, we get

$$\mathbb{P}(d(X_{(i-1)\cdot 2^{-n}}, X_{i\cdot 2^{-n}}) > 2^{-n\alpha}) \leq C2^{nq\alpha}2^{-(1+\epsilon)n}.$$

By summing over i we get

$$\begin{aligned}\mathbb{P}\left(\cup_{i=1}^{2^n}\{d(X_{(i-1)\cdot 2^{-n}}, X_{i\cdot 2^{-n}}) > 2^{-n\alpha}\}\right) &\leq 2^n \cdot C2^{nq\alpha}2^{-(1+\epsilon)n} \\ &= C2^{-n(\epsilon - q\alpha)}.\end{aligned}$$

Proof of Theorem 2.9 (cont)

Since $\epsilon - q\alpha > 0$, we have

$$\sum_{n=1}^{\infty} \mathbb{P}\left(\cup_{i=1}^{2^n}\{d(X_{(i-1)\cdot 2^{-n}}, X_{i\cdot 2^{-n}}) > 2^{-n\alpha}\}\right) < \infty,$$

and the Borel-Cantelli lemma implies that, with probability 1, we can find a finite integer $n_0(\omega)$ such that

$$d(X_{(i-1)\cdot 2^{-n}}, X_{i\cdot 2^{-n}}) \leq 2^{-n\alpha}, \quad \text{for all } n \geq n_0(\omega), i = 1, \ldots, 2^n.$$

Hence the constant

$$K_\alpha(\omega) = \sup_{n\geq 1}\left(\sup_{1\leq i\leq 2^n} \frac{d(X_{(i-1)\cdot 2^{-n}}, X_{i\cdot 2^{-n}})}{2^{-n\alpha}}\right)$$

is finite a.s.

Proof of Theorem 2.9 (cont)

Thus it follows from Lemma 2.10 that, on the event $\{K_\alpha(\omega) < \infty\}$, we have, for every $s, t \in D$,

$$d(X_s, X_t) \leq C_\alpha(\omega)|t - s|^\alpha$$

where $C_\alpha(\omega) = 2(1 - 2^{-\alpha})^{-1} K_\alpha$. Consequently, on the event $\{K_\alpha(\omega) < \infty\}$, the mapping $t \mapsto X_t(\omega)$ is Hölder continuous with exponent α on D, hence uniformly continuous on D. Since (E, d) is complete, this mapping has a unique continuous extension to $I = [0, 1]$, which is also Hölder continuous with exponent α. We can thus define, for every $t \in [0, 1]$,

$$\widetilde{X}_t(\omega) = \begin{cases} \lim_{D \ni s \to t} X_s(\omega) & \text{if } K_\alpha(\omega) < \infty \\ x_0 & \text{if } K_\alpha(\omega) = \infty \end{cases}$$

where x_0 is a fixed point in E. Clearly $\widetilde{X}_t$ is a random variable.

Proof of Theorem 2.9 (cont)

The sample paths of $\widetilde{X}$ is Hölder continuous with exponent α on $[0, 1]$. We now verify that $\widetilde{X}$ is a modification of X. For this, Fix $t \in [0, 1]$. The assumption of the theorem implies that

$$\lim_{s \to t} X_s = X_t, \quad \text{in probability.}$$

Since by definition $\widetilde{X}_t$ is the almost sure limit of X_s, when $D \ni s \to t$, we get that $\widetilde{X}_t = X_t$ a.s.

Corollary 2.11

Let $B = (B_t)_{t \geq 0}$ be a pre-Brownian motion. B has modification whose sample paths are continuous, even locally Hölder continuous with exponent $\frac{1}{2} - \delta$ for every $\delta \in (0, \frac{1}{2})$.

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Proof of Corollary 2.11

If $s < t$, then $B_t - B_s \stackrel{d}{=} \sqrt{t-s}U$, where U is a standard normal. Thus, for any $q > 0$,

$$\mathbb{E}[|B_t - B_s|^q] = (t-s)^{q/2}\mathbb{E}[|U|^q] \leq C_q|t-s|^{q/2}.$$

Taking $q > 2$ and applying Theorem 2.9 with $\epsilon = \frac{q}{2} - 1$ $(\alpha < (\frac{q}{2} - 1)/q = (q-2)/(2q))$.

Definition 2.12

A process $B = (B_t)_{t \geq 0}$ is called a Brownian motion if (i) B is a pre-Browian motion; (ii) all sample paths of B are continuous.

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This is the definition of a 1-dimensional Brownian motion starting from the origin. Extensions to arbitrary starting points and to higher dimensions will be discussed later.

The existence of Brownian motion in the sense of the preceding definition follows from Corollary 2.11. Indeed, starting from a pre-Brownian motion, this corollary provides a modification with continuous sample paths, which is still a pre-Brownian motion.

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