Math 562 Fall 2020

Renming Song

University of Illinois at Urbana-Champaign

August 31, 2020
Outline
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1. General Info
2. 2.1 Pre-Brownian motion
3. Continuity of Sample Paths
HW1 is posted on my homepage. I also setup HW1 in the course Moodle page. You need to submit your HW via Moodle. The due date for HW1 is 09/08 at noon.

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1. General Info
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Throughout this chapter, we work on a probability space \((\Omega, \mathcal{F}, P)\).

**Definition 2.1**

Let \(G\) be a Gaussian white noise on \(\mathbb{R}_+\) whose intensity is the Lebesgue measure. The process \((B_t)_{t \in \mathbb{R}_+}\) defined by

\[
B_t = G(1_{[0,t]})
\]

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Proposition 2.2

A pre-Brownian motion is a centered Gaussian process with covariance

\[ K(s, t) = \min\{s, t\} = s \land t. \]

Proof of Proposition 2.2

By definition, the random variables \( B_t \) belong to a common Gaussian space and therefore \( (B_t)_{t \geq 0} \) is a Gaussian process. Moreover, for \( s, t \geq 0 \),

\[
\mathbb{E}[B_s B_t] = \mathbb{E}[G([0, s]) G([0, t])] = \int_0^\infty 1_{[0,s]}(r) 1_{[0,t]}(r) \, dr = s \land t.
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Proposition 2.3

Let \((X_t)_{t \geq 0}\) be a real-valued process. The following are equivalent:

(i) \((X_t)_{t \geq 0}\) is a pre-Brownian motion;

(ii) \((X_t)_{t \geq 0}\) is a centered Gaussian process with covariance \(K(s, t) = s \wedge t\);

(iii) \(X_0 = 0\) a.s., and for any \(0 \leq s \leq t\), \(X_t - X_s\) is independent of \(\sigma(X_r : r \leq s)\) and \(X_t - X_s\) is \(\mathcal{N}(0, t - s)\);

(iv) \(X_0 = 0\) a.s., and for every choice \(0 = t_0 < t_1 \cdots < t_p\), \(X_{t_i} - X_{t_{i-1}}, 1 \leq i \leq p\), are indep, and for every \(1 \leq i \leq p\), \(X_{t_i} - X_{t_{i-1}}\) is \(\mathcal{N}(0, t_i - t_{i-1})\).
Proof of Proposition 2.3

(i) ⇒ (ii) follows from Proposition 2.2.
(ii) ⇒ (iii) Assume \((X_t)_{t\geq 0}\) is a centered Gaussian process with covariance \(K(s, t) = s \wedge t\) and that \(H\) is the Gaussian space generated by \((X_t)_{t\geq 0}\). Then \(X_0\) is a \(\mathcal{N}(0, 0)\) variable and hence \(X_0 = 0\) a.s. Fix \(s > 0\) and write \(H_s\) for the vector space spanned by \(\{X_r : 0 \leq r \leq s\}\), and \(\tilde{H}_s\) for the vector space spanned by \(\{X_{u+s} - X_s : u \geq 0\}\). Then \(H_s\) and \(\tilde{H}_s\) are orthogonal since, for \(r \in [0, s]\) and \(u \geq 0\),

\[
\mathbb{E}[X_r(X_{u+s} - X_s)] = r \wedge (s + u) - r \wedge s = r - r = 0.
\]

This implies that \(\sigma(H_s)\) and \(\sigma(\tilde{H}_s)\) are independent. In particular, if we fix \(t > s\), \(X_t - X_s\) is independent of \(\sigma(H_s) = \sigma(X_r : r \leq s)\). Finally, using the form of the covariance function, we immediately get that \(X_t - X_s\) is a \(\mathcal{N}(0, t - s)\) variable.
Proof of Proposition 2.3 (cont)

(iii) ⇒ (iv) is straightforward.
(iv) ⇒ (i) It follows from (iv) that \((X_t)_{t \geq 0}\) is a centered Gaussian process. If \(f\) is a step function on \(\mathbb{R}_+\) of the form \(f = \sum_{j=1}^{n} \lambda_j 1_{(t_{j-1}, t_j]}\), where \(0 = t_0 < t_1 < \cdots < t_n\), we set

\[
G(f) = \sum_{j=1}^{n} \lambda_j (X_{t_j} - X_{t_{j-1}}).
\]

Suppose that \(f\) and \(g\) are two step functions. We can write
\[
f = \sum_{j=1}^{n} \lambda_j 1_{(t_{j-1}, t_j]} \quad \text{and} \quad g = \sum_{j=1}^{n} \mu_j 1_{(t_{j-1}, t_j]} \quad \text{with} \quad 0 = t_0 < t_1 < \cdots < t_n.
\]
It then follows from a simple calculation that

\[
\mathbb{E}[G(f)G(g)] = \int_{\mathbb{R}_+} f(t)g(t) \, dt,
\]
so that \(G\) is an isometry from the vector space of step functions on \(\mathbb{R}_+\) into the Gaussian space \(H\) generated by \((X_t)_{t \geq 0}\).
Proof of Proposition 2.3 (cont)

Using the fact that step functions are dense in $L^2(\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+), dt)$, we get that the mapping $f \mapsto G(f)$ can be extended to an isometry from $L^2(\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+), dt)$ into the Gaussian space $H$. Finally, we have $G([0, t]) = X_t - X_0 = X_t$ by construction.

Corollary 2.4

Let $(B_t)_{t \geq 0}$ be a pre-Brownian motion. Then for every choice of $0 = t_0 < t_1 < \cdots < t_n$, the law of the vector $(B_{t_1}, B_{t_2}, \ldots, B_{t_n})$ has density

$$p(x_1, \ldots, x_n) = \frac{1}{(2\pi)^{n/2} \sqrt{t_1(t_2 - t_1) \cdots (t_n - t_{n-1})}} e^{-\sum_{j=1}^n \frac{(x_j-x_{j-1})^2}{2(t_j-t_{j-1})}}$$

where $x_0 = 0$ by convention.
Proof of Proposition 2.3 (cont)

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where $x_0 = 0$ by convention.
Proof of Corollary 2.4

The random variables $B_{t_1}, B_{t_2} - B_{t_1}, \ldots, B_{t_n} - B_{t_{n-1}}$ are independent and $B_{t_n} - B_{t_{n-1}}$ is $\mathcal{N}(0, t_j - t_{j-1})$. Hence the vector $(B_{t_1}, B_{t_2} - B_{t_1}, \ldots, B_{t_n} - B_{t_{n-1}})$ has density

$$q(y_1, \ldots, y_n) = \frac{1}{(2\pi)^{n/2} \sqrt{t_1(t_2 - t_1) \cdots (t_n - t_{n-1})}} e^{-\sum_{j=1}^{n} \frac{y_j^2}{2(t_j - t_{j-1})}}.$$ 

Now apply the change of variables $x_j = y_1 + \cdots + y_j, j = 1, \ldots, n$.

Remark

Corollary 2.4, together with the property $B_0 = 0$, determines the collection of finite-dimensional marginal distributions of pre-Brownian motion. Proposition 2.3.(iv) shows that a process having the same finite-dimensional marginal distributions as pre-Brownian motion must also be a pre-Brownian motion.
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Proposition 2.5

Let \((B_t)_{t \geq 0}\) be a pre-Brownian motion. Then

(i) \((-B_t)_{t \geq 0}\) is a pre-Brownian motion;

(ii) for every \(\lambda > 0\), \((\frac{1}{\lambda} B_{\lambda^2 t})_{t \geq 0}\) is a pre-Brownian motion;

(iii) for every \(s \geq 0\), \((B_{s+t} - B_s)_{t \geq 0}\) is a pre-Brownian motion and is independent of \(\sigma(B_r : r \leq s)\).

Proof of Proposition 2.5

(i) and (ii) are easy. (iii) With the notation of the proof of Proposition 2.3, the \(\sigma\)-field generated by \((B_{s+t} - B_s)_{t \geq 0}\) is \(\sigma(\tilde{H}_s)\), which is independent of \(\sigma(H_s) = \sigma(X_r : r \leq s)\). To see that \((B_{s+t} - B_s)_{t \geq 0}\) is a pre-Brownian motion it suffices to verify property (iv) of Prop 2.3, which is immediate.
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Let \((B_t)_{t \geq 0}\) be a pre-Brownian motion and let \(G\) be the associated Gaussian white noise. Note that \(G\) is determined by \((B_t)_{t \geq 0}\). If \(f\) is a step function there is an explicit formula for \(G(f)\) in terms of \((B_t)_{t \geq 0}\), and one then uses a density argument. One often writes for \(f \in L^2(\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+), dt)\),

\[
G(f) = \int_0^\infty f(s) dB_s,
\]

and similarly

\[
G(f1_{[0,t]}) = \int_0^t f(s) dB_s, \quad G(f1_{(s,t]}) = \int_s^t f(r) dB_r.
\]

This notation is justified by the fact that, if \(u < v\),

\[
\int_u^v dB_s = G((u, v]) = G([0, v]) - G([0, u]) = B_v - B_u.
\]
The mapping $f \mapsto \int_0^\infty f(s)dB_s$ (that is, the Gaussian white noise $G$) is then called the Wiener integral with respect to $B$. Note that $\int_0^\infty f(s)dB_s$ is a $\mathcal{N}(0, \int_0^\infty f^2(s)ds)$ variable.

Since a Gaussian white noise is not a genuine measure depending on $\omega$, $\int_0^\infty f(s)dB_s$ is not a genuine integral depending on $\omega$. 
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Let $E$ be a metric space equipped with its Borel $\sigma$-field $\mathcal{B}(E)$.

**Definition 2.6**

Let $(X_t)_{t \in T}$ be an $E$-valued process. The sample paths of $X$ are the mappings $T \ni t \mapsto X_t(\omega)$ obtained when fixing $\omega \in \Omega$. The sample paths of $X$ thus form a collection of mappings from $T$ to $E$ indexed by $\omega \in \Omega$.

For a pre-Brownian motion $B = (B_t)_{t \geq 0}$, we have no info about the sample paths. The sample paths may not even be measurable functions. At the cost of “slightly” modifying $B$, we can ensure that sample paths are continuous.
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Definition 2.7

Let \((X_t)_{t \in T}\) and \((\tilde{X}_t)_{t \in T}\) be 2 processes indexed by the same index set \(T\), with values in the same metric space \(E\) and defined on the same probability space \((\Omega, \mathcal{F}, \mathbb{P})\). We say that \(\tilde{X}\) is a modification or version of \(X\) if

\[
\mathbb{P}(X_t = \tilde{X}_t) = 1, \quad \text{for all } t \in T.
\]

This implies that \(\tilde{X}\) has the same finite-dimensional marginals as \(X\). Thus, if \(X\) is a pre-Brownian motion, \(\tilde{X}\) is also a pre-Brownian motion. On the other hand, sample paths of \(\tilde{X}\) may have very different properties from those of \(X\).
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Definition 2.8

Let \((X_t)_{t \in T}\) and \((\tilde{X}_t)_{t \in T}\) be 2 processes indexed by the same index set \(T\), with values in the same metric space \(E\) and defined on the same probability space \((\Omega, \mathcal{F}, P)\). \(\tilde{X}\) is said to be indistinguishable from \(X\) if there exists a negligible subset \(N\) of \(\Omega\) such that

\[
\tilde{X}_t(\omega) = X_t(\omega), \quad \text{for all } \omega \in \Omega \setminus N \text{ and } t \in T.
\]

If \(\tilde{X}\) is indistinguishable from \(X\), then \(\tilde{X}\) is a modification of \(X\). Indistinguishability is a much stronger notion. Two indistinguishable processes has a.s. the same sample paths. We always identity 2 indistinguishable processes. “Unique” always means “unique up to indistinguishability”.
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If $\tilde{X}$ is indistinguishable from $X$, then $\tilde{X}$ is a modification of $X$. Indistinguishability is a much stronger notion. Two indistinguishable processes has a.s. the same sample paths. We always identity 2 indistinguishable processes. “Unique” always means “unique up to indistinguishability”.
Suppose $T = I$ is an interval of $\mathbb{R}$. If the sample paths of both $X$ and $\tilde{X}$ are continuous (right continuous, or left-continuous), except possibly on a negligible subset of $\Omega$, then $\tilde{X}$ is indistinguishable from $X$ if and only if $\tilde{X}$ is a modification of $X$.

**Theorem 2.9 (Kolmogorov’s continuity theorem)**

Let $X = (X_t)_{t \in I}$ be a process indexed by a bounded interval of $\mathbb{R}$ and taking values in a complete metric space $(E, d)$. Assume that there exist three reals $q, \epsilon, C$ such that, for every $s, t \in I$,

$$\mathbb{E}[d(X_s, X_t)^q] \leq C|t - s|^{1+\epsilon}.$$

Then, there is a modification $\tilde{X}$ of $X$ whose sample paths are Hölder continuous with exponent $\alpha$ for every $\alpha \in (0, \frac{\epsilon}{q})$. This means that, for
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Every \( \omega \in \Omega \) and \( \alpha \in (0, \frac{\epsilon}{q}) \), there exists a finite constant \( C_\alpha(\omega) \) such that, for every \( s, t \in I \),

\[
d(\tilde{X}_s(\omega), \tilde{X}_t(\omega)) \leq C_\alpha(\omega)|t - s|^\alpha.
\]

In particular, \( \tilde{X} \) a modification of \( X \) with continuous sample paths. Such a modification is unique up to indistinguishability.
Remarks

(i) If $I$ is unbounded, we get from this theorem that $X$ has a modification whose sample paths are locally Hölder with exponent $\alpha$ for every $\alpha \in (0, \frac{\epsilon}{q})$.

(ii) It suffices to prove that, for every fixed $\alpha \in (0, \frac{\epsilon}{q})$, $X$ has a modification whose sample paths are Hölder with exponent $\alpha$. Indeed, we can then apply this result to every choice of $\alpha$ in a sequence $\alpha_k \uparrow \frac{\epsilon}{q}$, noting that the resulting modifications are indistinguishable.