Math 562 Fall 2020

Renming Song

University of Illinois at Urbana-Champaign

August 31, 2020

Outline

Outline

HW1 is posted on my homepage. I also setup HW1 in the course Moodle page. You need to submit your HW via Moodle. The due date for HW1 is 09/08 at noon.

Slides and videos of the lectures are also available from the Moodle page.

HW1 is posted on my homepage. I also setup HW1 in the course Moodle page. You need to submit your HW via Moodle. The due date for HW1 is 09/08 at noon.

Slides and videos of the lectures are also available from the Moodle page.

Outline

Throughout this chapter, we work on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

Definition 2.1

Let G be a Gaussian white noise on $\mathbb{R}_+$ whose intensity is the Lebesgue measure. The process $(B_t)_{t\in\mathbb{R}_+}$ defined by

$$B_t = G(1_{[0,t]})$$

is called a pre-Brownian motion.

Throughout this chapter, we work on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

Definition 2.1

Let G be a Gaussian white noise on $\mathbb{R}_+$ whose intensity is the Lebesgue measure. The process $(B_t)_{t\in\mathbb{R}_+}$ defined by

$$B_t = G(1_{[0,t]})$$

is called a pre-Brownian motion.

Proposition 2.2

A pre-Brownian motion is a centered Gaussian process with covariance

$$K(s,t) = \min\{s,t\} = s \wedge t.$$

Proof of Proposition 2.2

By definition, the random variables B_t belong to a common Gauussian space and therefore $(B_t)_{t\geq 0}$ is a Gaussian process. Moreover, for $s, t \geq 0$,

$$\begin{aligned}\mathbb{E}[B_s B_t] &= \mathbb{E}[G([0,s])G([0,t])] \\ &= \int_0^\infty 1_{[0,s]}(r)1_{[0,t]}(r)dr = s \wedge t.\end{aligned}$$

Proposition 2.2

A pre-Brownian motion is a centered Gaussian process with covariance
$$K(s,t) = \min\{s,t\} = s \wedge t.$$

Proof of Proposition 2.2

By definition, the random variables B_t belong to a common Gauussian space and therefore $(B_t)_{t\geq 0}$ is a Gaussian process. Moreover, for $s, t \geq 0$,

$$\begin{aligned}\mathbb{E}[B_s B_t] &= \mathbb{E}[G([0,s])G([0,t])] \\ &= \int_0^\infty 1_{[0,s]}(r) 1_{[0,t]}(r) dr = s \wedge t.\end{aligned}$$

Proposition 2.3

Let $(X_t)_{t\geq 0}$ be a real-valued process. The following are equivalent:

(i) $(X_t)_{t\geq 0}$ is a pre-Brownian motion;

(ii) $(X_t)_{t\geq 0}$ is a centered Gaussian process with covariance $K(s,t) = s \wedge t$;

(iii) $X_0 = 0$ a.s., and for any $0 \leq s \leq t$, $X_t - X_s$ is independent of $\sigma(X_r : r \leq s)$ and $X_t - X_s$ is $\mathcal{N}(0, t-s)$;

(iv) $X_0 = 0$ a.s., and for every choice $0 = t_0 < t_1 \cdots < t_p$, $X_{t_i} - X_{t_{i-1}}$, $1 \leq i \leq p$, are indep, and for for every $1 \leq i \leq p$, $X_{t_i} - X_{t_{i-1}}$ is $\mathcal{N}(0, t_i - t_{i-1})$.

Proof of Proposition 2.3

(i) $\Rightarrow$ (ii) follows from Proposition 2.2.
(ii) $\Rightarrow$ (iii) Assume $(X_t)_{t\geq 0}$ is a centered Gaussian process with covariance $K(s,t) = s \wedge t$ and that H is the Gaussian space generated by $(X_t)_{t\geq 0}$. Then X_0 is a $\mathcal{N}(0,0)$ variable and hence $X_0 = 0$ a.s. Fix $s > 0$ and write H_s for the vector space spanned by $\{X_r : 0 \leq r \leq s\}$, and $\widetilde{H}_s$ for the vector space spanned by $\{X_{u+s} - X_s : u \geq 0\}$. Then H_s and $\widetilde{H}_s$ are orthogonal since, for $r \in [0,s]$ and $u \geq 0$,

$$\mathbb{E}[X_r(X_{u+s} - X_s)] = r \wedge (s+u) - r \wedge s = r - r = 0.$$

This implies that $\sigma(H_s)$ and $\sigma(\widetilde{H}_s)$ are independent. In particular, if we fix $t > s$, $X_t - X_s$ is independent of $\sigma(H_s) = \sigma(X_r : r \leq s)$. Finally, using the form of the covariance function, we immediately get that $X_t - X_s$ is a $\mathcal{N}(0, t-s)$ variable.

Proof of Proposition 2.3 (cont)

(iii) $\Rightarrow$ (iv) is straightforward.
(iv) $\Rightarrow$ (i) It follows from (iv) that $(X_t)_{t\geq 0}$ is a centered Gaussian process. If f is a step function on $\mathbb{R}_+$ of the form $f = \sum_{j=1}^n \lambda_j 1_{(t_{j-1},t_j]}$, where $0 = t_0 < t_1 < \cdots < t_n$, we set

$$G(f) = \sum_{j=1}^{n} \lambda_j (X_{t_j} - X_{t_{j-1}}).$$

Suppose that f and g are two step functions. We can write $f = \sum_{j=1}^n \lambda_j 1_{(t_{j-1},t_j]}$ and $g = \sum_{j=1}^n \mu_j 1_{(t_{j-1},t_j]}$ with $0 = t_0 < t_1 < \cdots < t_n$. It then follows from a simple calculation that

$$\mathbb{E}[G(f)G(g)] = \int_{\mathbb{R}_+} f(t)g(t)dt,$$

so that G is an isometry from the vector space of step functions on $\mathbb{R}_+$ into the Gaussian space H generated by $(X_t)_{t\geq 0}$.

Proof of Proposition 2.3 (cont)

Using the fact that step functions are dense in $L^2(\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+), dt)$, we get that the mapping $f \mapsto G(f)$ can be extended to an isometry from $L^2(\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+), dt)$ into the Gaussian space H. Finally, we have $G([0, t]) = X_t - X_0 = X_t$ by construction.

Corollary 2.4

Let $(B_t)_{t\geq 0}$ be a pre-Brownian motion. Then for every choice of $0 = t_0 < t_1 < \cdots < t_n$, the law of the vector $(B_{t_1}, B_{t_2}, \ldots, B_{t_n})$ has density

$$p(x_1, \ldots, x_n) = \frac{1}{(2\pi)^{n/2}\sqrt{t_1(t_2 - t_1)\cdots(t_n - t_{n-1})}} e^{-\sum_{j=1}^{n} \frac{(x_j - x_{j-1})^2}{2(t_j - t_{j-1})}}$$

where $x_0 = 0$ by convention.

Proof of Proposition 2.3 (cont)

Using the fact that step functions are dense in $L^2(\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+), dt)$, we get that the mapping $f \mapsto G(f)$ can be extended to an isometry from $L^2(\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+), dt)$ into the Gaussian space H. Finally, we have $G([0,t]) = X_t - X_0 = X_t$ by construction.

Corollary 2.4

Let $(B_t)_{t\geq 0}$ be a pre-Brownian motion. Then for every choice of $0 = t_0 < t_1 < \cdots < t_n$, the law of the vector $(B_{t_1}, B_{t_2}, \ldots, B_{t_n})$ has density

$$p(x_1, \ldots, x_n) = \frac{1}{(2\pi)^{n/2}\sqrt{t_1(t_2 - t_1)\cdots(t_n - t_{n-1})}} e^{-\sum_{j=1}^{n} \frac{(x_j - x_{j-1})^2}{2(t_j - t_{j-1})}}$$

where $x_0 = 0$ by convention.

Proof of Corollary 2.4

The random variables $B_{t_1}, B_{t_2} - B_{t_1}, \ldots, B_{t_n} - B_{t_{n-1}}$ are independent and $B_{t_n} - B_{t_{n-1}}$ is $\mathcal{N}(0, t_j - t_{j-1})$. Hence the vector $(B_{t_1}, B_{t_2} - B_{t_1}, \ldots, B_{t_n} - B_{t_{n-1}})$ has density

$$q(y_1, \ldots, y_n) = \frac{1}{(2\pi)^{n/2}\sqrt{t_1(t_2 - t_1)\cdots(t_n - t_{n-1})}} e^{-\sum_{j=1}^n \frac{y_j^2}{2(t_j - t_{j-1})}}.$$

Now apply the the change of variables $x_j = y_1 + \cdots + y_j$, $j = 1, \ldots, n$.

Remark

Corollary 2.4, together with the property $B_0 = 0$, determines the collection of finite-dimensional marginal distributions of pre-Brownian motion. Proposition 2.3.(iv) shows that a process having the same finite-dimensional marginal distributions as pre-Brownian motion must also be a pre-Brownian motion.

Proof of Corollary 2.4

The random variables $B_{t_1}, B_{t_2} - B_{t_1}, \ldots, B_{t_n} - B_{t_{n-1}}$ are independent and $B_{t_n} - B_{t_{n-1}}$ is $\mathcal{N}(0, t_j - t_{j-1})$. Hence the vector $(B_{t_1}, B_{t_2} - B_{t_1}, \ldots, B_{t_n} - B_{t_{n-1}})$ has density

$$q(y_1, \ldots, y_n) = \frac{1}{(2\pi)^{n/2}\sqrt{t_1(t_2 - t_1)\cdots(t_n - t_{n-1})}} e^{-\sum_{j=1}^{n} \frac{y_j^2}{2(t_j - t_{j-1})}}.$$

Now apply the the change of variables $x_j = y_1 + \cdots + y_j$, $j = 1, \ldots, n$.

Remark

Corollary 2.4, together with the property $B_0 = 0$, determines the collection of finite-dimensional marginal distributions of pre-Brownian motion. Proposition 2.3.(iv) shows that a process having the same finite-dimensional marginal distributions as pre-Brownian motion must also be a pre-Brownian motion.

Proposition 2.5

Let $(B_t)_{t\geq 0}$ be a pre-Brownian motion. Then

(i) $(-B_t)_{t\geq 0}$ is a pre-Brownian motion;

(ii) for every $\lambda > 0$, $(\frac{1}{\lambda}B_{\lambda^2 t})_{t\geq 0}$ is a pre-Brownian motion;

(iii) for every $s \geq 0$, $(B_{s+t} - B_s)_{t\geq 0}$ is a a pre-Brownian motion and is independent of $\sigma(B_r : r \leq s)$.

Proof of Proposition 2.5

(i) and (ii) are easy. (iii) With the notation of the proof of Proposition 2.3, the σ-field generated by $(B_{s+t} - B_s)_{t\geq 0}$ is $\sigma(\widetilde{H}_s)$, which is independent of $\sigma(H_s) = \sigma(X_r : r \leq s)$. To see that $(B_{s+t} - B_s)_{t\geq 0}$ is a pre-Brownian motion it suffices to verify property (iv) of Prop 2.3, which is immediate.

Proposition 2.5

Let $(B_t)_{t\geq 0}$ be a pre-Brownian motion. Then

(i) $(-B_t)_{t\geq 0}$ is a pre-Brownian motion;

(ii) for every $\lambda > 0$, $(\frac{1}{\lambda} B_{\lambda^2 t})_{t\geq 0}$ is a pre-Brownian motion;

(iii) for every $s \geq 0$, $(B_{s+t} - B_s)_{t\geq 0}$ is a a pre-Brownian motion and is independent of $\sigma(B_r : r \leq s)$.

Proof of Proposition 2.5

(i) and (ii) are easy. (iii) With the notation of the proof of Proposition 2.3, the σ-field generated by $(B_{s+t} - B_s)_{t\geq 0}$ is $\sigma(\widetilde{H}_s)$, which is independent of $\sigma(H_s) = \sigma(X_r : r \leq s)$. To see that $(B_{s+t} - B_s)_{t\geq 0}$ is a pre-Brownian motion it suffices to verify property (iv) of Prop 2.3, which is immediate.

Let $(B_t)_{t\geq 0}$ be a pre-Brownian motion and let G be the associated Gaussian white noise. Note that G is determined by $(B_t)_{t\geq 0}$. If f is a step function there is an explicit formula for $G(f)$ in terms of $(B_t)_{t\geq 0}$, and one then uses a density argument. One often writes for $f \in L^2(\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+), dt)$,

$$G(f) = \int_0^\infty f(s)dB_s,$$

and similarly

$$G(f1_{[0,t]}) = \int_0^t f(s)dB_s, \quad G(f1_{(s,t]}) = \int_s^t f(r)dB_r.$$

This notation is justified by the fact that, if $u < v$,

$$\int_u^v dB_s = G((u,v]) = G([0,v]) - G([0,u]) = B_v - B_u.$$

The mapping $f \mapsto \int_0^\infty f(s)dB_s$ (that is, the Gaussian white noise G) is then called the Wiener integral with respect to B. Note that $\int_0^\infty f(s)dB_s$ is a $\mathcal{N}(0, \int_0^\infty f^2(s)ds)$ variable.

Since a Gaussian white noise is not a genuine measure depending on ω, $\int_0^\infty f(s)dB_s$ is not a genuine integral depending on ω.

The mapping $f \mapsto \int_0^\infty f(s)dB_s$ (that is, the Gaussian white noise G) is then called the Wiener integral with respect to B. Note that $\int_0^\infty f(s)dB_s$ is a $\mathcal{N}(0, \int_0^\infty f^2(s)ds)$ variable.

Since a Gaussian white noise is not a genuine measure depending on ω, $\int_0^\infty f(s)dB_s$ is not a genuine integral depending on ω.

Outline

Let E be a metric space equipped with its Borel σ-filed $\mathcal{B}(E)$.

Definition 2.6

Let $(X_t)_{t \in T}$ be an E-valued process. The sample paths of X are the mappings $T \ni t \mapsto X_t(\omega)$ obtained when fixing $\omega \in \Omega$. The sample paths of X thus form a collection of mappings from T to E indexed by $\omega \in \Omega$.

For a pre-Brownian motion $B = (B_t)_{t \geq 0}$, we have no info about the sample paths. The sample paths may not even be measurable functions. At the cost of"slightly" modifying B, we can ensure that sample paths are continuous.

Let E be a metric space equipped with its Borel σ-filed $\mathcal{B}(E)$.

Definition 2.6

Let $(X_t)_{t \in T}$ be an E-valued process. The sample paths of X are the mappings $T \ni t \mapsto X_t(\omega)$ obtained when fixing $\omega \in \Omega$. The sample paths of X thus form a collection of mappings from T to E indexed by $\omega \in \Omega$.

For a pre-Brownian motion $B = (B_t)_{t \geq 0}$, we have no info about the sample paths. The sample paths may not even be measurable functions. At the cost of"slightly" modifying B, we can ensure that sample paths are continuous.

Let E be a metric space equipped with its Borel σ-filed $\mathcal{B}(E)$.

Definition 2.6

Let $(X_t)_{t \in T}$ be an E-valued process. The sample paths of X are the mappings $T \ni t \mapsto X_t(\omega)$ obtained when fixing $\omega \in \Omega$. The sample paths of X thus form a collection of mappings from T to E indexed by $\omega \in \Omega$.

For a pre-Brownian motion $B = (B_t)_{t \geq 0}$, we have no info about the sample paths. The sample paths may not even be measurable functions. At the cost of"slightly" modifying B, we can ensure that sample paths are continuous.

Definition 2.7

Let $(X_t)_{t\in T}$ and $(\widetilde{X}_t)_{t\in T}$ be 2 processes indexed by the same index set T, with values in the same metric space E and defined on the same probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We say that $\widetilde{X}$ is a modification or version of X if

$$\mathbb{P}(X_t = \widetilde{X}_t) = 1, \quad \text{for all } t \in T.$$

This implies that $\widetilde{X}$ has the same finite-dimensional marginals as X. Thus, if X is a pre-Brownian motion, $\widetilde{X}$ is also a pre-Brownian motion. On the other hand, sample paths of $\widetilde{X}$ may have very different properties from those of X.

Definition 2.7

Let $(X_t)_{t\in T}$ and $(\widetilde{X}_t)_{t\in T}$ be 2 processes indexed by the same index set T, with values in the same metric space E and defined on the same probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We say that $\widetilde{X}$ is a modification or version of X if

$$\mathbb{P}(X_t = \widetilde{X}_t) = 1, \quad \text{for all } t \in T.$$

This implies that $\widetilde{X}$ has the same finite-dimensional marginals as X. Thus, if X is a pre-Brownian motion, $\widetilde{X}$ is also a pre-Brownian motion. On the other hand, sample paths of $\widetilde{X}$ may have very different properties from those of X.

Definition 2.8

Let $(X_t)_{t \in T}$ and $(\widetilde{X}_t)_{t \in T}$ be 2 processes indexed by the same index set T, with values in the same metric space E and defined on the same probability space $(\Omega, \mathcal{F}, \mathbb{P})$. $\widetilde{X}$ is said to be indistinguishable from X if there exists a negligible subset N of Ω such that

$$\widetilde{X}_t(\omega) = X_t(\omega), \quad \text{for all } \omega \in \Omega \setminus N \text{ and } t \in T.$$

If $\widetilde{X}$ is indistinguishable from X, then $\widetilde{X}$ is a modification of X. Indistinguishability is a much stronger notion. Two indistinguishable processes has a.s. the same sample paths. We always identity 2 indistinguishable processes. “Unique” always means “unique up to indistinguishability”.

Definition 2.8

Let $(X_t)_{t\in T}$ and $(\widetilde{X}_t)_{t\in T}$ be 2 processes indexed by the same index set T, with values in the same metric space E and defined on the same probability space $(\Omega, \mathcal{F}, \mathbb{P})$. $\widetilde{X}$ is said to be indistinguishable from X if there exists a negligible subset N of Ω such that

$$\widetilde{X}_t(\omega) = X_t(\omega), \qquad \text{for all } \omega \in \Omega \setminus N \text{ and } t \in T.$$

If $\widetilde{X}$ is indistinguishable from X, then $\widetilde{X}$ is a modification of X. Indistinguishability is a much stronger notion. Two indistinguishable processes has a.s. the same sample paths. We always identity 2 indistinguishable processes. “Unique” always means “unique up to indistinguishability”.

Suppose $T = I$ is an interval of $\mathbb{R}$. If the sample paths of both X and $\widetilde{X}$ are continuous (right continuous, or left-continuous), except possibly on a negligible subset of Ω, then $\widetilde{X}$ is indistinguishable from X if and only if $\widetilde{X}$ is a modification of X.

Theorem 2.9 (Kolmogorov's continuity theorem)

Let $X = (X_t)_{t \in I}$ be a process indexed by a bounded interval of $\mathbb{R}$ and taking values in a complete metric space (E, d). Assume that there exist three reals q, ϵ, C such that, for every $s, t \in I$,

$$\mathbb{E}[d(X_s, X_t)^q] \leq C|t - s|^{1+\epsilon}.$$

Then, there is a modification $\widetilde{X}$ of X whose sample paths are Hölder continuous with exponent α for every $\alpha \in (0, \frac{\epsilon}{q})$. This means that, for

Suppose $T = I$ is an interval of $\mathbb{R}$. If the sample paths of both X and $\widetilde{X}$ are continuous (right continuous, or left-continuous), except possibly on a negligible subset of Ω, then $\widetilde{X}$ is indistinguishable from X if and only if $\widetilde{X}$ is a modification of X.

Theorem 2.9 (Kolmogorov's continuity theorem)

Let $X = (X_t)_{t \in I}$ be a process indexed by a bounded interval of $\mathbb{R}$ and taking values in a complete metric space (E, d). Assume that there exist three reals q, ϵ, C such that, for every $s, t \in I$,

$$\mathbb{E}[d(X_s, X_t)^q] \leq C|t - s|^{1+\epsilon}.$$

Then, there is a modification $\widetilde{X}$ of X whose sample paths are Hölder continuous with exponent α for every $\alpha \in (0, \frac{\epsilon}{q})$. This means that, for

Theorem 2.9 (Kolmogorov's continuity theorem)

every $\omega \in \Omega$ and $\alpha \in (0, \frac{\epsilon}{q})$, there exists a finite constant $C_\alpha(\omega)$ such that, for every $s, t \in I$,

$$d(\widetilde{X}_s(\omega), \widetilde{X}_t(\omega)) \leq C_\alpha(\omega)|t - s|^\alpha.$$

In particular, $\widetilde{X}$ a modification of X with continuous sample paths. Ssuch a modification is unique up to indistinguishability.

Remarks

(i) If I is unbounded, we get from this theorem that X has a modification whose sample paths are locally Hölder with exponent α for every $\alpha \in (0, \frac{\epsilon}{q})$.
(ii) It suffices to prove that, for every fixed $\alpha \in (0, \frac{\epsilon}{q})$, X has a modification whose sample paths are H"older with exponent α. Indeed, we can then apply this result to every choice of α in a sequence $\alpha_k \uparrow \frac{\epsilon}{q}$, noting that the resulting modifications are indistinguishable.