Math 562 Fall 2020

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Outline

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You will submit your homework there also. I have not created the Assignments yet.

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Let $(X_t)_{t\in T}$ be a (centered) Gaussian process. The covariance function of $(X_t)_{t\in T}$ is the function $\Gamma : T \times T \mapsto \mathbb{R}$ defined by

$$\Gamma(s,t) = \mathrm{Cov}(X_s, X_t) = \mathbb{E}[X_s X_t].$$

This function characterizes the collection of finite-dimensional marginal distributions of the process $(X_t)_{t\in T}$, that is, the collection consisting for every choice of the distinct indices $t_1, \ldots, t_p$ in T, $p \geq 1$, of the law of the vector $(X_{t_1}, \ldots X_{t_p})$. Indeed this vector is a centered R^p-valued Gaussian vector in with covariance matrix $(\Gamma(t_i, t_j))_{1\leq i,j\leq p}$.

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Given a function Γ on $T \times T$, one may ask whether there exists a Gaussian process $(X_t)_{t\in T}$ whose covariance function in Γ. The function Γ must be symmetric ($\Gamma(s,t) = \Gamma(t,s)$) and of positive type in the following sense: if c is a real-valued function on T with finite support, then

$$\sum_{T\times T} c(s)c(t)\Gamma(s,t) \geq 0.$$

Indeed, if Γ is the covariance function of the process $(X_t)_{t\in T}$, we must have

$$\sum_{T\times T} c(s)c(t)\Gamma(s,t) = \mathrm{Var}\left(\sum_{T} c(s)X_s\right) \geq 0.$$

The next theorem solves the existence problem in the general case. This theorem is a direct consequence of the Kolmogorov extension theorem. We omit the proof.

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Theorem 1.11

Let Γ be a symmetric function of positive type on $T \times T$. There exists, on an appropriate probability space $(\Omega, \mathcal{F}, \mathbb{P})$, a centered Gaussian process whose covariance function is Γ.

Example

Consider the case $T = \mathbb{R}$ and let μ be a finite symmetric Borel measure on $\mathbb{R}$ ($\mu(-A) = \mu(A)$ for any Borel subset A of $\mathbb{R}$). Define, for $s, t \in \mathbb{R}$,

$$\Gamma(s,t) = \int e^{i\xi(t-s)} \mu(d\xi) \in \mathbb{R}.$$

Γ is symmetric. If c is a real-valued function on $\mathbb{R}$ with finite support,

$$\sum_{\mathbb{R}\times\mathbb{R}} c(s)c(T)\Gamma(s,t) = \int \left|\sum_{\mathbb{R}} c(s)e^{i\xi s}\right|^2 \mu(d\xi) \geq 0.$$

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Example (cont)

The function Γ has the additional property that $\Gamma(s,t)$ only depends on $t-s$. It immediately follows that any (centered) Gaussian process $(X_t)_{t\in\mathbb{R}}$ with covariance function is stationary (in a strong sense), meaning that

$$(X_{t_1+t}, X_{t_2+t}, \dots, X_{t_n+t}) \stackrel{d}{=} (X_{t_1}, X_{t_2}, \dots, X_{t_n})$$

for any choice of $t_1, t_2, \dots, t_n, t \in \mathbb{R}$. Conversely, any stationary Gaussian process $(X_t)_{t\in\mathbb{R}}$ has a covariance function of the preceding type, and the measure μ is called the spectral measure of $(X_t)_{t\in\mathbb{R}}$.

Outline

Definition 1.12

Let $(E, \mathcal{E})$ be a measurable space, and let μ be a σ-finite measure on $(E, \mathcal{E})$. A Gaussian white noise with intensity μ is an isometry G from $L^2(E, \mathcal{E}, \mu)$ into a (centered) Gaussian space.

Hence, if $f \in L^2(E, \mathcal{E}, \mu)$, $G(f)$ is centered Gaussian with variance

$$\mathbb{E}[G(f)^2] = \|f\|^2_{L^2(E,\mathcal{E},\mu)} = \int f^2 d\mu.$$

If $f, g \in L^2(E, \mathcal{E}, \mu)$, the covariance of $G(f)$ and $G(g)$ is

$$\mathbb{E}[G(f)G(g)] = \langle f, g \rangle_{L^2(E,\mathcal{E},\mu)} = \int fg d\mu.$$

In particular, if $f = 1_A$ with $\mu(A) < \infty$, then $G(1_A)$ is a $\mathcal{N}(0, \mu(A))$ variable, we will write $G(A) = G(1_A)$.

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In particular, if $f = 1_A$ with $\mu(A) < \infty$, then $G(1_A)$ is a $\mathcal{N}(0, \mu(A))$ variable, we will write $G(A) = G(1_A)$.

Let $A_1, \dots, A_n \in \mathcal{E}$ be disjoint with $\mu(A_j) < \infty$ for all $j = 1, \dots, n$. Then the vector

$$(G(A_1), \dots, G(A_n))$$

is a Gaussian vector in $\mathbb{R}^n$ and its covariance matrix is diagonal since,if $i \neq j$,

$$\mathbb{E}[G(A_i)G(A_j)] = \langle 1_{A_i}, 1_{A_j} \rangle_{L^2(E,\mathcal{E},\mu)} = 0.$$

Suppose $A \in \mathcal{E}$ with $\mu(A) < \infty$ and that A is the disjoint union of a countable collection $A_1, A_2, \dots$ of elemenst of $\mathcal{E}$. Then $1_A = \sum_{j=1}^{\infty} 1_{A_j}$ in $L^2(E, \mathcal{E}, \mu)$ and so by the isometry property

$$G(A) = \sum_{j=1}^{\infty} G(A_j), \text{ in } L^2(E, \mathcal{E}, \mu).$$

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Properties of the mapping $A \mapsto G(A)$ are therefore very similar to those of a measure depending on the parameter $\omega \in \Omega$. However, one can show that, if ω is fixed, the mapping $A \mapsto G(A)(\omega)$ does not (in general) define a measure.

Proposition 1.13

Let $(E, \mathcal{E})$ be a measurable space, and let μ be a σ-finite measure on $(E, \mathcal{E})$. There exists, on an appropriate probability space $(\Omega, \mathcal{F}, \mathbb{P})$, a Gaussian white noise with intensity μ.

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Proof of Proposition 1.13

Take $T = L^2(E, \mathcal{E}, \mu)$ and $\Gamma(f, g) = \langle f, g \rangle_{L^2(E,\mathcal{E},\mu))}$. Then Γ satisfies the conditions of Theorem 1.11 (is symmetric and of positive type). Thus there exists a Gaussian process $(X_f, f \in L^2(E, \mathcal{E}, \mu))$. Just take $G(f) = X_f$.

Another proof of Proposition 1.13 (cont)

Let $(f_i, i \in I)$ be a complete orthonormal system in the Hilbert space $L^2(E, \mathcal{E}, \mu)$. For every $f \in L^2(E, \mathcal{E}, \mu)$,

$$f = \sum_{i \in I} \alpha_i f_i$$

with $\alpha_i = \langle f, f_i \rangle$ satisfying

$$\sum_{i \in I} \alpha_i^2 = \|f\|^2.$$

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Proof of Proposition 1.13 (cont)

On an appropriate probability space $(\Omega, \mathcal{F}, \mathbb{P})$, we can construct a collection $(X_i)_{i \in I}$ of independent $\mathcal{N}(0,1)$ random variables, and we define

$$G(f) = \sum_{i \in I} \alpha_i X_i.$$

The series converges in L^2 since $(X_i)_{i \in I}$ form an orthonormal system in L^2. Then clearly G takes values in the Gaussian space generated by $(X_i)_{i \in I}$. Furthermore, G is an isometry since it maps the orthonormal basis $(f_i, i \in I)$ to an orthonormal system.

In what follows, we will only consider the case when $L^2(E, \mathcal{E}, \mu)$ is separable. For instance, if $(E, \mathcal{E}) = (\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+))$ and μ is the Lebesgue measure, the construction of G only requires a sequence $(\xi_n)_{n\geq 0}$ of independent $\mathcal{N}(0,1)$ random variables, and the choice of an orthonormal basis $(\varphi_n)_{n\geq 0}$ of $L^2(\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+), dt)$. We get G by setting

$$G(f) = \sum_{n\geq 0} \langle f, \varphi_n \rangle \xi_n.$$

The next result gives a way of recovering the intensity $\mu(A)$ of a measurable set A from the values of G.

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Proposition 1.14

Let G be a Gaussian white noise on $(E, \mathcal{E})$ with intensity μ. Let $A \in \mathcal{E}$ with $\mu(A) < \infty$. Assume that there exists a sequence of partitions of A

$$A = A_1^n \cup \cdots \cup^n_{k_n}$$

whose "mesh" tends to 0 on the sense that

$$\lim_{n \to \infty} \left(\sup_{1 \leq j \leq k_n} \mu(A_j^n) \right) = 0.$$

Then

$$\lim_{n \to \infty} \sum_{j=1}^{k_n} G(A_j^n)^2 = \mu(A)$$

in L^2.

Proof of Proposition 1.14

For every fixed n, $G(A_1^n), \dots, G(A_{k_n}^n)$ are independent. Furthermore, $\mathbb{E}[G(A_j^n)^2] = \mu(A_j^n)$. Thus

$$\mathbb{E}\left[\left(\sum_{j=1}^{k_n} G(A_j^n)^2 - \mu(A)\right)^2\right] = \sum_{j=1}^{k_n} \mathrm{Var}(G(A_j^n)^2) = 2\sum_{j=1}^{k_n} \mu(A_j^n)^2,$$

because, if X is an $\mathcal{N}(0, \sigma^2)$ random variable, $\mathrm{Var}(X^2) = \mathbb{E}(X^4) - \sigma^4 = 3\sigma^4 - \sigma^4 = 2\sigma^4$. Then

$$\sum_{j=1}^{k_n} \mu(A_j^n)^2 \leq \left(\sup_{1\leq j\leq k_n} \mu(A_j^n)\right) \mu(A) \to 0$$

by assumption.

Outline

Throughout this chapter, we work on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

Definition 2.1

Let G be a Gaussian white noise on $\mathbb{R}_+$ whose intensity is the Lebesgue measure. The process $(B_t)_{t \in \mathbb{R}_+}$ defined by

$$B_t = G(1_{[0,t]})$$

is called a pre-Brownian motion.

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Proposition 2.2

A pre-Brownian motion is a centered Gaussian process with covariance

$$K(s,t) = \min\{s,t\} = s \wedge t.$$

Proof of Proposition 2.2

By definition, the random variables B_t belong to a common Gauussian space and therefore $(B_t)_{t\geq 0}$ is a Gaussian process. Moreover, for $s, t \geq 0$,

$$\begin{aligned}\mathbb{E}[B_s B_t] &= \mathbb{E}[G([0,s])G([0,t])] \\ &= \int_0^\infty 1_{[0,s]}(r) 1_{[0,t]}(r) dr = s \wedge t.\end{aligned}$$

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Proposition 2.3

Let $(X_t)_{t\geq 0}$ be a real-valued process. The following are equivalent:

(i) $(X_t)_{t\geq 0}$ is a pre-Brownian motion;

(ii) $(X_t)_{t\geq 0}$ is a centered Gaussian process with covariance $K(s,t) = s \wedge t$;

(iii) $X_0 = 0$ a.s., and for any $0 \leq s \leq t$, $X_t - X_s$ is independent of $\sigma(X_r : r \leq s)$ and $X_t - X_s$ is $\mathcal{N}(0, t-s)$;

(iv) $X_0 = 0$ a.s., and for every choice $0 = t_0 < t_1 \cdots < t_p$, $X_{t_i} - X_{t_{i-1}}, 1 \leq i \leq p$, are indep, and for for every $1 \leq i \leq p$, $X_{t_i} - X_{t_{i-1}}$ is $\mathcal{N}(0, t_i - t_{i-1})$.