Math 562 Fall 2020

Renming Song

University of Illinois at Urbana-Champaign

August 24, 2020

Outline

Outline

1. Course Info
2. Gaussian Random variables
3. Gaussian vectors

Course Syllabus is available from my homepage:
https://faculty.math.illinois.edu/ rsong/562f20/562f20.html

Textbook: Jean-Francois Le Gall: Brownian Motion, Martingales, and Stochastic Calculus, Springer, 2016. Our library has an electronic copy.

Office hours (via Zoom): MWF noon -12:50 pm

Your grade for the course will depend on homework assignment and a take-home final exam.

Course Syllabus is available from my homepage:
https://faculty.math.illinois.edu/ rsong/562f20/562f20.html

Textbook: Jean-Francois Le Gall: Brownian Motion, Martingales, and Stochastic Calculus, Springer, 2016. Our library has an electronic copy.

Office hours (via Zoom): MWF noon -12:50 pm

Your grade for the course will depend on homework assignment and a take-home final exam.

Course Syllabus is available from my homepage:
https://faculty.math.illinois.edu/ rsong/562f20/562f20.html

Textbook: Jean-Francois Le Gall: Brownian Motion, Martingales, and Stochastic Calculus, Springer, 2016. Our library has an electronic copy.

Office hours (via Zoom): MWF noon -12:50 pm

Your grade for the course will depend on homework assignment and a take-home final exam.

Course Syllabus is available from my homepage:
https://faculty.math.illinois.edu/ rsong/562f20/562f20.html

Textbook: Jean-Francois Le Gall: Brownian Motion, Martingales, and Stochastic Calculus, Springer, 2016. Our library has an electronic copy.

Office hours (via Zoom): MWF noon -12:50 pm

Your grade for the course will depend on homework assignment and a take-home final exam.

Course topics: Gaussian processes, martingales, stochastic integration, Markov processes and stochastic differential equations.

Gaussian processes, Markov processes and martingales are three of the most important classes of stochastic processes. Brownian motion belongs to all three classes.

We start with Gaussian processes.

Course topics: Gaussian processes, martingales, stochastic integration, Markov processes and stochastic differential equations.

Gaussian processes, Markov processes and martingales are three of the most important classes of stochastic processes. Brownian motion belongs to all three classes.

We start with Gaussian processes.

Course topics: Gaussian processes, martingales, stochastic integration, Markov processes and stochastic differential equations.

Gaussian processes, Markov processes and martingales are three of the most important classes of stochastic processes. Brownian motion belongs to all three classes.

We start with Gaussian processes.

Course topics: Gaussian processes, martingales, stochastic integration, Markov processes and stochastic differential equations.

Gaussian processes, Markov processes and martingales are three of the most important classes of stochastic processes. Brownian motion belongs to all three classes.

We start with Gaussian processes.

Outline

A real-valued random variable X is said to be a standard *Gaussian* (or *normal*) random variable if it has density

$$p_X(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right), \quad x \in \mathbb{R}$$

with respect to the Lebesgue measure on $\mathbb{R}$.

We have

$$\mathbb{E}\left[e^{zX}\right] = e^{z^2/2}, \quad z \in \mathbb{C}.$$

($z \in \mathbb{R}$ easy, then using analytic continuation.) In particular,

$$\mathbb{E}\left[e^{i\xi X}\right] = e^{-\xi^2/2}, \quad \xi \in \mathbb{R}.$$

A real-valued random variable X is said to be a standard *Gaussian* (or *normal*) random variable if it has density

$$p_X(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right), \quad x \in \mathbb{R}$$

with respect to the Lebesgue measure on $\mathbb{R}$.

We have

$$\mathbb{E}\left[e^{zX}\right] = e^{z^2/2}, \quad z \in \mathbb{C}.$$

($z \in \mathbb{R}$ easy, then using analytic continuation.) In particular,

$$\mathbb{E}\left[e^{i\xi X}\right] = e^{-\xi^2/2}, \quad \xi \in \mathbb{R}.$$

From the expansion

$$\mathbb{E}\left[e^{i\xi X}\right] = 1 + i\xi\mathbb{E}[X] + \cdots + \frac{(i\xi)^n}{n!}\mathbb{E}[X^n] + O(|\xi|^{n+1})$$

as $\xi \to 0$, we get

$$\mathbb{E}[X] = 0; \quad \mathbb{E}[X^2] = 1,$$

and

$$\mathbb{E}[X^{2n}] = \frac{(2n)!}{2^n n!}, \quad \mathbb{E}[X^{2n+1}] = 0, \quad n \geq 1.$$

If $\sigma > 0$ and $m \in \mathbb{R}$, we say that a real-valued random variable Y is Gaussian with $\mathcal{N}(m, \sigma^2)$-distribution if Y satisfies any of the 3 equivalent properties below:

(i) $Y = \sigma X + m$, where X is a standard Gaussian random variable;

(ii) Y has density

$$p_Y(y) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(y-m)^2}{2\sigma^2}\right), \quad y \in \mathbb{R};$$

(iii) the characteristic function of Y is

$$\mathbb{E}[e^{i\xi Y}] = e^{im\xi - \frac{\sigma^2}{2}\xi^2}, \quad \xi \in R.$$

If Y is a Gaussian random variable with $\mathcal{N}(m, \sigma^2)$-distribution, then

$$\mathbb{E}[Y] = m, \quad \mathrm{Var}(Y) = \sigma^2.$$

If $\sigma > 0$ and $m \in \mathbb{R}$, we say that a real-valued random variable Y is Gaussian with $\mathcal{N}(m, \sigma^2)$-distribution if Y satisfies any of the 3 equivalent properties below:

(i) $Y = \sigma X + m$, where X is a standard Gaussian random variable;

(ii) Y has density

$$p_Y(y) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(y-m)^2}{2\sigma^2}\right), \quad y \in \mathbb{R};$$

(iii) the characteristic function of Y is

$$\mathbb{E}[e^{i\xi Y}] = e^{im\xi - \frac{\sigma^2}{2}\xi^2}, \quad \xi \in R.$$

If Y is a Gaussian random variable with $\mathcal{N}(m, \sigma^2)$-distribution, then

$$\mathbb{E}[Y] = m, \quad \mathrm{Var}(Y) = \sigma^2.$$

We say that Y is a Gaussian random variable with $\mathcal{N}(m, 0)$-distribution if $Y = m$ a.s.

It is well-known that if Y_1 is a Gaussian random variable with $\mathcal{N}(m_1, \sigma_1^2)$-distribution, Y_2 is a Gaussian random variable with $\mathcal{N}(m_2, \sigma_2^2)$-distribution, and Y_1 and Y_2 are independent, then $Y_1 + Y_2$ is a Gaussian random variable with $\mathcal{N}(m_1 + m_2, \sigma_1^2 + \sigma_2^2)$-distribution.

We say that Y is a Gaussian random variable with $\mathcal{N}(m, 0)$-distribution if $Y = m$ a.s.

It is well-known that if Y_1 is a Gaussian random variable with $\mathcal{N}(m_1, \sigma_1^2)$-distribution, Y_2 is a Gaussian random variable with $\mathcal{N}(m_2, \sigma_2^2)$-distribution, and Y_1 and Y_2 are independent, then $Y_1 + Y_2$ is a Gaussian random variable with $\mathcal{N}(m_1 + m_2, \sigma_1^2 + \sigma_2^2)$-distribution.

Proposition 1.1

Let $(X_n)_{n\geq 1}$ be a sequence of real-valued random variables such that, for every $n \geq 1$, X_n follows the $\mathcal{N}(m_n, \sigma_n^2)$-distribution. Suppose that X_n converges in L^2 to X. Then

(i) The random variable X follows the $\mathcal{N}(m, \sigma^2)$-distribution, with $m = \lim_{n\to\infty} m_n$ and $\sigma^2 = \lim_{n\to\infty} \sigma_n^2$;

(ii) the convergence also holds in L^p for all $p \in [1, \infty)$.

Proof of Proposition 1.1

(i) The convergence in L^2 implies that $m_n = \mathbb{E}[X_n] \to \mathbb{E}[X]$ and $\sigma_n^2 = \text{Var}(X_n) \to \text{Var}(X)$. Then setting $m = \mathbb{E}[X]$ and $\sigma^2 = \text{Var}(X)$, we get, for any $\xi \in \mathbb{R}$,

$$\mathbb{E}[e^{i\xi X}] = \lim_{n\to\infty} \mathbb{E}[e^{i\xi X_n}] = \lim_{n\to\infty} e^{im_n\xi - \frac{\sigma_n^2}{2}\xi^2} = e^{im\xi - \frac{\sigma^2}{2}\xi^2}.$$

Thus X is a Gaussian random variable with the $\mathcal{N}(m, \sigma^2)$-distribution.

Proof of Proposition 1.1 (cont)

(ii) Since $X_n = \sigma_n N + m_n$ with N a standard Gaussian random variable, and since $(m_n)_{n\geq 1}$ and $(\sigma_n^2)_{n\geq 1}$ are bounded, we have that, for any $q \geq 1$,

$$\sup_n \mathbb{E}[|X_n|^q] < \infty.$$

It follows that, for any $q \geq 1$,

$$\sup_n \mathbb{E}[|X_n - X|^q] < \infty.$$

Let $p \in [1, \infty)$. The sequence $Y_n = |X_n - X|^p$ converges to 0 in probability and is uniformly integrable because it is bounded in L^2. It follows that this sequence converges to 0 converges to 0 in L^1, which is the desired result.

Outline

Let E be a d-dimensional vector space. We write $\langle u, v\rangle$ for the inner product in E. An E-valued random variable X is called a *Gaussian vector* if, for every $u \in E$, $\langle u, X\rangle$ is a (real-valued) Gaussian variable.

Let X be an E-valued Gaussian vector. Then there exist $m_X \in E$ and a non-negative quadratic form q_X on E such that, for any $u \in E$,

$$\mathbb{E}[\langle u, X\rangle] = \langle u, m_X\rangle, \quad \mathrm{Var}(\langle u, X\rangle) = q_X(u).$$

Indeed, let $(e_1, \ldots, e_d)$ be an orthonormal basis of E and write $X = \sum_{j=1}^{d} X_j e_j$ in this basis. Note that $X_j = \langle e_J, X\rangle$ is Gaussian for every j. It is then immediate that the preceding formulas hold with $m_X = \sum_{j=1}^{d} \mathbb{E}[X_j] e_j =: \mathbb{E}[X]$ and, if $u = \sum_{j=1}^{d} u_j e_j$,

$$q_X(u) = \sum_{j,k=1}^{d} u_j u_k \mathrm{Cov}(X_j, X_k).$$

Let E be a d-dimensional vector space. We write $\langle u, v\rangle$ for the inner product in E. An E-valued random variable X is called a *Gaussian vector* if, for every $u \in E$, $\langle u, X\rangle$ is a (real-valued) Gaussian variable.

Let X be an E-valued Gaussian vector. Then there exist $m_X \in E$ and a non-negative quadratic form q_X on E such that, for any $u \in E$,

$$\mathbb{E}[\langle u, X\rangle] = \langle u, m_X\rangle, \quad \mathrm{Var}(\langle u, X\rangle) = q_X(u).$$

Indeed, let $(e_1, \ldots, e_d)$ be an orthonormal basis of E and write $X = \sum_{j=1}^d X_j e_j$ in this basis. Note that $X_j = \langle e_J, X\rangle$ is Gaussian for every j. It is then immediate that the preceding formulas hold with $m_X = \sum_{j=1}^d \mathbb{E}[X_j] e_j =: \mathbb{E}[X]$ and, if $u = \sum_{j=1}^d u_j e_j$,

$$q_X(u) = \sum_{j,k=1}^d u_j u_k \mathrm{Cov}(X_j, X_k).$$

Let E be a d-dimensional vector space. We write $\langle u, v\rangle$ for the inner product in E. An E-valued random variable X is called a *Gaussian vector* if, for every $u \in E$, $\langle u, X\rangle$ is a (real-valued) Gaussian variable.

Let X be an E-valued Gaussian vector. Then there exist $m_X \in E$ and a non-negative quadratic form q_X on E such that, for any $u \in E$,

$$\mathbb{E}[\langle u, X\rangle] = \langle u, m_X\rangle, \quad \mathrm{Var}(\langle u, X\rangle) = q_X(u).$$

Indeed, let $(e_1, \ldots, e_d)$ be an orthonormal basis of E and write $X = \sum_{j=1}^d X_j e_j$ in this basis. Note that $X_j = \langle e_J, X\rangle$ is Gaussian for every j. It is then immediate that the preceding formulas hold with $m_X = \sum_{j=1}^d \mathbb{E}[X_j]e_j =: \mathbb{E}[X]$ and, if $u = \sum_{j=1}^d u_j e_j$,

$$q_X(u) = \sum_{j,k=1}^d u_j u_k \mathrm{Cov}(X_j, X_k).$$

Since $\langle u, X\rangle$ is Gaussian with $\mathcal{N}(\langle u, m_X\rangle, q_X(u))$-distribution, we get

$$\mathbb{E}[e^{i\langle u,X\rangle}] = e^{i\langle u,m_X\rangle - \frac{1}{2}q_X(u)}, \quad u \in E.$$

Proposition 1.2

Under the preceding assumptions, the random variables $X_1, \ldots, X_d$ are independent if and only if the covariance matrix $(\mathrm{Cov}(X_j, X_k))_{1\le j,k\le d}$ is diagonal, or equivalently if and only if q_X is of diagonal form in the basis $(e_1, \ldots, e_d)$.

Since $\langle u, X\rangle$ is Gaussian with $\mathcal{N}(\langle u, m_X\rangle, q_X(u))$-distribution, we get

$$\mathbb{E}[e^{i\langle u,X\rangle}] = e^{i\langle u,m_X\rangle - \frac{1}{2}q_X(u)}, \quad u \in E.$$

Proposition 1.2

Under the preceding assumptions, the random variables $X_1, \ldots, X_d$ are independent if and only if the covariance matrix $(\mathrm{Cov}(X_j, X_k))_{1\le j,k\le d}$ is diagonal, or equivalently if and only if q_X is of diagonal form in the basis $(e_1, \ldots, e_d)$.

Proof of Proposition 1.2

If the random variables $X_1, \ldots, X_d$ are independent, then the covariance matrix $(\mathrm{Cov}(X_j, X_k))_{1 \le j,k \le d}$ is diagonal. Conversely, if this matrix is diagonal, we have for every $u = \sum_{j=1}^{d} u_j e_j \in E$,

$$q_X(u) = \sum_{j=1}^{d} \lambda_j u_j^2,$$

where $\lambda_j = \mathrm{Var}(X_j)$. Consequently,

$$\mathbb{E}[e^{i \sum_{j=1}^{d} u_j X_j}] = \prod_{j=1}^{d} e^{iu_j \mathbb{E}[X_j] - \frac{1}{2}\lambda_j u_j^2} = \prod_{j=1}^{d} \mathbb{E}[e^{iu_j X_j}].$$

which implies that $X_1, \ldots, X_d$ are independent.

With the quadratic form q_X, we associate the unique symmetric endomorphism γ_X of E such that

$$q_X(u) = \langle u, \gamma_X(u) \rangle.$$

Note that γ_X is non-negative in the sense that its eigenvalues are all non-negative.

From now on, to simplify the statements, we restrict our attention to centered Gaussian vectors, i.e. such that $m_X = 0$, but the results are easily adapted to the non-centered case.

With the quadratic form q_X, we associate the unique symmetric endomorphism γ_X of E such that

$$q_X(u) = \langle u, \gamma_X(u) \rangle.$$

Note that γ_X is non-negative in the sense that its eigenvalues are all non-negative.

From now on, to simplify the statements, we restrict our attention to centered Gaussian vectors, i.e. such that $m_X = 0$, but the results are easily adapted to the non-centered case.

Theorem 1.3

(i) Let γ be a non-negative symmetric endomorphism of E. Then there exists a Gaussian vector X such that $\gamma_X = \gamma$.
(ii) Let X be a centered Gaussian vector. Let $(\epsilon_1, \ldots, \epsilon_d)$ be a basis of E in which γ_X is diagonal, $\gamma_X(\epsilon_j) = \lambda_j \epsilon_j$ for every $j = 1, \ldots d$, where

$$\lambda_1 \geq \lambda_2 \geq \cdots \lambda_r > 0 = \lambda_{r+1} = \cdots = \lambda_d$$

so that r is the rank of γ_X. Then

$$X = \sum_{j=1}^{r} Y_j \epsilon_j,$$

where $Y_j, j = 1, \ldots, r$, are independent (centered) Gaussian variables and the variance of Y_j is λ_j. Consequently, if P_X denotes the distribution of X, the topological support of P_X is the vector space spanned by $\epsilon_1, \ldots, \epsilon_r$.

Theorem 1.3 (cont)

Furthermore, P_X is absolutely continuous with respect to the Lebesgue measure on E if and only if $r = d$, and in that case the density of X is

$$p_X(x) = \frac{1}{(2\pi)^{d/2}\sqrt{\det(\gamma_X)}} e^{-\frac{1}{2}\langle x, \gamma_X^{-1}(x)\rangle}.$$

Proof of Theorem 1.3

(i) Let $(\epsilon_1, \dots, \epsilon_d)$ be an orthonormal basis of E in which γ is diagonal, $\gamma(\epsilon_j) = \lambda_j \epsilon_j$, $j = 1, \dots d$, and let $Y_1, \dots, Y_d$ be independent centered Gaussian variables with $\mathrm{Var}(Y_j) = \lambda_j, j = 1, \dots d$. We set

$$X = \sum_{j=1}^{d} Y_j \epsilon_j.$$

Then, if $u = \sum_{j=1}^{d} u_j \epsilon_j$,

$$q_X(u) = \mathbb{E}\left[(\sum_{j=1}^{d} u_j Y_j)^2 \right] = \sum_{j=1}^{d} \lambda_j u_j^2 = \langle u, \gamma(u) \rangle.$$

Proof of Theorem 1.3 (cont)

(ii) Let $Y_1, \dots, Y_d$ be the coordinates of X in the basis $(\epsilon_1, \dots, \epsilon_d)$. Then the matrix of γ_X in this basis is the covariance matrix of $(Y_1, \dots, Y_d)$. The covariance matrix is diagonal and, by Proposition 1.2, the random variables $Y_1, \dots, Y_d$ are independent. Furthermore, for $j \in \{r+1, \dots, d\}$, $\mathbb{E}[Y_j^2] = 0$ and hence $Y_j = 0$ a.s.
Then, since $X = \sum_{j=1}^r Y_j \epsilon_j$ a.s., it is clear that $\mathrm{Supp}(P_X)$ is contained in the the subspace spanned by $\epsilon_1, \dots, \epsilon_r$. Conversely, if

$$O = \{u = \sum_{j=1}^r \alpha_j \epsilon_j, a_j < \alpha_j < b_j, j = 1, \dots, r\},$$

we have $\mathbb{P}[X \in O] = \prod_{j=1}^r \mathbb{P}[a_j < \alpha_j < b_j] > 0$. This implies that $\mathrm{Supp}(P_X)$ is the subspace spanned by $\epsilon_1, \dots, \epsilon_r$.

Proof of Theorem 1.3 (cont)

If $r < d$, the vector space spanned by $\epsilon_1, \dots, \epsilon_r$ has zero Lebesgue measure, thus distribution of X is singular with respect to Lebesgue measure on E. Suppose that $r = d$, and let $Y = (Y_1, \dots, Y_d) \in \mathbb{R}^d$. Note that the bijection $\varphi(y_1, \dots, y_d) = \sum_{j=1}^d y_j \epsilon_j$ maps Y to X. Then writing $y = (y_1, \dots, y_d)$, we have

$$\begin{aligned}
\mathbb{E}[g(X)] =& \mathbb{E}[g(\varphi(Y))] \\
=& \frac{1}{(2\pi)^{1/2}} \int_{\mathbb{R}^d} g(\varphi(y)) e^{-\frac{1}{2}\sum_{j=1}^d \frac{y_j^2}{\lambda_j}} \frac{dy_1 \cdots dy_d}{\sqrt{\lambda_1 \cdots \lambda_d}}, \quad \text{indep} \\
=& \frac{1}{(2\pi)^{1/2}\sqrt{\det(\gamma_X)}} \int_{\mathbb{R}^d} g(\varphi(y)) e^{-\frac{1}{2}\langle \varphi(y), \gamma_X^{-1}(\varphi(y))\rangle} dy_1 \cdots dy_d \\
=& \frac{1}{(2\pi)^{1/2}\sqrt{\det(\gamma_X)}} \int_E g(x) e^{-\frac{1}{2}\langle x, \gamma_X^{-1}(x)\rangle} dx
\end{aligned}$$

Proof of Theorem 1.3 (cont)

since Lebesgue measure on E is by definition the image of Lebesgue measure on $\mathbb{R}^d$ under φ. In the third equality above we use that

$$\langle \varphi(y), \gamma_X^{-1}(\varphi(y)) \rangle = \langle \sum_{j=1}^{d} y_j \epsilon_j, \sum_{j=1}^{d} \frac{y_j}{\lambda_j} \epsilon_j \rangle.$$