Math 561 Midterm Test

1. (25 points) Suppose that $X_1, X_2, \ldots$ are uncorrelated random variables with $\mathbb{E}(X_j) = \mu_j$ and $\text{Var}(X_j)/j \to 0$ as $j \to \infty$. Let $S_n = X_1 + \cdots + X_n$ and $\nu_n = \mathbb{E}(S_n)/n$. Show that as $n \to \infty$,

$$\frac{S_n}{n} - \nu_n \to 0,$$

in probability.
2. (25 points) Suppose that $X_1, X_2, \ldots$ are independent random variables. For all positive integers $m < n$, let $S_{m,n} = X_{m+1} + \cdots + X_n$. Show that, for any $a > 0$,

$$
P(\max_{m < j \leq n} |S_{m,j}| > 2a) \min_{m < k \leq n} P(|S_{k,n}| \leq a) \leq P(|S_{m,n}| > a).$$


3. (25 points) Suppose that \( X_1, X_2, \ldots \) are independent and identically distributed random variables with

\[
\mathbb{P}(X_1 = 2^j) = 2^{-j}, \quad j = 1, 2, \ldots
\]

Use the second Borel-Cantelli lemma to show that

\[
\limsup_{n \to \infty} \frac{X_n}{n \log_2 n} = \infty
\]

almost surely.
4. (25 points) Suppose that $X_1, X_2, \ldots$ are uncorrelated and identically distributed random variables with $\mathbb{E}(X_1) = 0$ and $\mathbb{E}(X_1^4) < \infty$. Let $S_n = X_1 + \cdots + X_n$. Use the Borel-Cantelli lemma to prove that $S_n/n$ converges to 0 almost surely.