Math 561, Spring 2022

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University of Illinois at Urbana-Champaign

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Outline
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1. Course Info

2. 1.1 Probability spaces
Course syllabus is available from my homepage: https://faculty.math.illinois.edu/~rsong/561s22/561s222.html


You do need a copy of this book. Most of the homework assignment will be from this book.

Office Hours: MWF: noon-12:50 pm in 227 CAB. I will also be on Zoom during this time.
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The first few lectures will be a quick review/summary of basic abstract measure theory. For those of you who are not familiar with abstract measure theory, you need to spend some extra time to catch up. We will use these in later chapters.

40% of your grade will depend on homework assignment, 30% will depend on the midterm test and 30% on the take-home final exam.

The Midterm test is on Friday, March 11 in our regular classroom during class time. The take-home final exam is due on May 6.
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1 Course Info

2 1.1 Probability spaces
Let $\Omega$ be a non-empty set. A collection $\mathcal{F}$ of subsets of $\Omega$ is called a $\sigma$-field (or $\sigma$-algebra) if

(i) $\Omega \in \mathcal{F}$;
(ii) $A \in \mathcal{F} \Rightarrow A^c \in \mathcal{F}$;
(iii) $A_i \in \mathcal{F}, i = 1, 2, \cdots \Rightarrow \bigcup_{i=1}^{\infty} A_i \in \mathcal{F}$.

If $\mathcal{F}$ is a $\sigma$-field of subsets of $\Omega$, then

(i) $\mathcal{F}$ is closed under finite union;
(ii) $\mathcal{F}$ is closed under (finite or countable) intersection.
Let $\Omega$ be a non-empty set. A collection $\mathcal{F}$ of subsets of $\Omega$ is called a \textit{\(\sigma\)-field (or \(\sigma\)-algebra)} if

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If $\mathcal{F}$ is a \(\sigma\)-field of subsets of $\Omega$, then

(i) $\mathcal{F}$ is closed under finite union;

(ii) $\mathcal{F}$ is closed under (finite or countable) intersection.
Given any collection \( C \) of subsets of \( \Omega \), there is a *smallest \( \sigma \)-field \( \mathcal{F} \) containing \( C \), that is, there is a \( \sigma \)-field \( \mathcal{F} \) containing \( C \) such that if \( \mathcal{F}' \) is any \( \sigma \)-field \( \mathcal{F} \) containing \( C \), then \( \mathcal{F}' \supset \mathcal{F} \).

The smallest \( \sigma \)-field \( \mathcal{F} \) containing \( C \) is called the \( \sigma \)-field generated by \( C \) and usually denoted by \( \sigma(\mathcal{C}) \).

A \( \sigma \)-field \( \mathcal{F} \) is said to be *countably generated* if there is a countable subcollection \( \mathcal{C} \subset \mathcal{F} \) such that \( \mathcal{F} = \sigma(\mathcal{C}) \).
Given any collection $\mathcal{C}$ of subsets of $\Omega$, there is a \textit{smallest $\sigma$-field $\mathcal{F}$ containing $\mathcal{C}$}, that is, there is a $\sigma$-field $\mathcal{F}$ containing $\mathcal{C}$ such that if $\mathcal{F}'$ is any $\sigma$-field $\mathcal{F}$ containing $\mathcal{C}$, then $\mathcal{F}' \supset \mathcal{F}$.

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Example

Let $\mathbb{R}^d$ be the $d$-dim Euclidean space. The smallest $\sigma$-field containing all open sets of $\mathbb{R}^d$ is called the *Borel $\sigma$-field* on $\mathbb{R}^d$ and is denoted by $\mathcal{B}^d$. When $d = 1$, we simply write $\mathcal{B}$.

$\mathcal{B}^d$ is countably generated since $\mathcal{B}^d$ is generated by

$$\{(a_1, b_1] \times \cdots \times (a_d, b_d] : a_i < b_i, a_i, b_i \text{ rational }, i = 1, \ldots, d\}.$$
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By a *measurable space* we mean a couple \((\Omega, \mathcal{F})\), where \(\Omega\) is a non-empty set and \(\mathcal{F}\) is a \(\sigma\)-field of subsets of \(\Omega\).

By a *measure* \(\mu\) on a measurable space \((\Omega, \mathcal{F})\), we mean a \([0, \infty]\)-valued function \(\mu\) on \(\mathcal{F}\) such that

(i) \(\mu(\emptyset) = 0\);

(ii) \(E_i \in \mathcal{F}, i = 1, 2, \ldots, \{E_i\} \text{ disjoint} \Rightarrow
\[
\mu \left( \bigcup_{i=1}^{\infty} E_i \right) = \sum_{i=1}^{\infty} \mu(E_i).
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If \(\mu(\Omega) = 1\), we say that \(\mu\) is a *probability measure*. We usually denote a probability measure by \(P\).
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Theorem 1.1.1

Let $\mu$ be a measure on $(\Omega, \mathcal{F})$. Then

(i) $A \subset B, A, B \in \mathcal{F} \Rightarrow \mu(A) \leq \mu(B)$;

(ii) $A, A_i \in \mathcal{F}, i = 1, 2, \ldots, A \subset \bigcup_{i=1}^{\infty} A_i \Rightarrow \mu(A) \leq \sum_{i=1}^{\infty} \mu(A_i)$;

(iii) $A, A_i \in \mathcal{F}, i = 1, 2, \ldots, A_i \uparrow A \Rightarrow \mu(A_i) \uparrow \mu(A)$;

(iv) $A, A_i \in \mathcal{F}, i = 1, 2, \ldots, A_i \downarrow A, \mu(A_1) < \infty \Rightarrow \mu(A_i) \downarrow \mu(A)$.

By a measure space we mean a triple $(\Omega, \mathcal{F}, \mu)$, where $(\Omega, \mathcal{F})$ is a measurable space and $\mu$ is a measure on $(\Omega, \mathcal{F})$.

A probability space is simply a measure space $(\Omega, \mathcal{F}, P)$ with $P(\Omega) = 1$. $\Omega$ is called the sample space (the set of possible outcomes), $\mathcal{F}$ is the set of “events”.
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Discrete probability space

Let $\Omega$ be a countable set. Let $\mathcal{F}$ be the collection of all subsets of $\Omega$. Let

$$P(A) = \sum_{\omega \in A} p(\omega), \quad A \in \mathcal{F},$$

where $p(\omega) \geq 0$ and $\sum_{\omega \in \Omega} p(\omega) = 1$.

In many cases when $\Omega$ be a finite set, we have $p(\omega) = 1/|\Omega|$. In these cases, $P$ is called a *uniform* probability on $\Omega$. 
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Measures on $\mathbb{R}$

Measures on $(\mathbb{R}, \mathcal{R})$ are defined via Stieltjes functions, i.e., functions with the following properties:

(i) $F$ is non-decreasing;
(ii) $F$ is right-continuous.

Theorem 1.1.4

Associated with each Stieltjes function $F$, there is a unique measure $\mu$ on $(\mathbb{R}, \mathcal{R})$ with

$$\mu((a, b]) = F(b) - F(a), \quad \text{for all } a \leq b.$$ 

I am not going to prove this theorem. See the appendix for the full proof. I will only give the main idea.
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A non-empty collection $S$ of subsets of $\Omega$ is called a *semi-algebra* if

(i) $S, T \in S \Rightarrow S \cap T \in S$;

(ii) $S \in S \Rightarrow S^c$ is a finite disjoint union of sets in $S$.

A non-empty collection $A$ of subsets of $\Omega$ is called an *algebra* (or *field*) if

(i) $A, B \in A \Rightarrow A \cup B \in A$;

(ii) $A \in A \Rightarrow A^c \in A$.

**Example**

$$S_d = \{(a_1, b_1] \times \cdots \times (a_d, b_d] : -\infty \leq a_i \leq b_i \leq \infty, i = 1, \ldots, d\}$$

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Lemma 1.1.7
If $S$ is a semi-algebra, then $\overline{S} = \{\text{finite disjoint unions of sets in } S\}$ is an algebra, called the \textit{algebra generated by} $S$.

Proof
Let $A = +_{i=1}^n S_i$ and $B = +_{j=1}^m T_j$ with $S_i, T_j \in S$, where $+_{i=1}^n$ denote disjoint union. Then $A \cap B = +_{ij}(S_i \cap T_j) \in \overline{S}$. If $A = +_{i=1}^n S_i$ with $S_i \in S$, then $A^c = \cap_i S_i^c$. By the definition of semi-algebra, $S_i^c \in \overline{S}$. We have shown that $\overline{S}$ is closed under intersection, so it follows by induction that $A^c \in \overline{S}$.

Example
If $S = S_1$, then

$$\overline{S} = \{\bigcup_{i=1}^k (a_i, b_i) : k = 1, 2, \ldots, -\infty \leq a_i \leq b_i \leq \infty, i = 1, \ldots, k\}.$$
Lemma 1.1.7

If $S$ is a semi-algebra, then $\overline{S} = \{ \text{finite disjoint unions of sets in } S \}$ is an algebra, called the *algebra generated by* $S$.

Proof

Let $A = \bigoplus_{i=1}^{n} S_i$ and $B = \bigoplus_{j=1}^{m} T_j$ with $S_i, T_j \in S$, where $\bigoplus_{i=1}^{n}$ denote disjoint union. Then $A \cap B = \bigoplus_{ij}(S_i \cap T_j) \in \overline{S}$. If $A = \bigoplus_{i=1}^{n} S_i$ with $S_i \in S$, then $A^c = \bigcap_i S_i^c$. By the definition of semi-algebra, $S_i^c \in \overline{S}$. We have shown that $\overline{S}$ is closed under intersection, so it follows by induction that $A^c \in \overline{S}$.

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Given a non-negative function $\mu$ on a semi-algebra $S$, we extend it to $S$ by

$$\mu(\bigcup_{i=1}^{n} A_i) = \sum_{i=1}^{n} \mu(A_i).$$

By a measure on an algebra $A$, we mean a function $\mu$ on $A$ such that

(i) $\mu(A) \geq \mu(\emptyset) = 0$;
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$\mu$ is said to be $\sigma$-finite if there is a sequence $A_n \in A$, $n = 1, 2, \ldots$ such that $\bigcup_{n=1}^{\infty} A_i = \Omega$ and $\mu(A_n) < \infty$ for each $n$. 
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The next result helps to extend a measure defined on a semi-algebra $S$ to the $\sigma$-algebra $\sigma(S)$.

**Theorem 1.1.9**

Let $S$ be a semi-algebra and let $\mu$ be defined on $S$ with $\mu(\emptyset) = 0$. Suppose (i) if $S \in S$ is a finite disjoint union of sets $A_i \in S$, then $\mu(S) = \sum_i \mu(A_i)$; (ii) If $S_i, S \in S$ with $S = \bigcup_{i=1}^{\infty} S_i$, then $\mu(S) \leq \sum_{i=1}^{\infty} \mu(S_i)$. Then $\mu$ has a unique extension $\bar{\mu}$ which is a measure on $\mathcal{S}$. If $\bar{\mu}$ is $\sigma$-finite, then there is a unique extension $\nu$ that is a measure on $\sigma(S)$.

Using this theorem, one can easily prove Theorem 1.1.4. We will do this next time.
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