Sketch of solutions to HW9

Chapter 6

9. Since the death rate is constant, it follows that as long as the system is nonempty, the number of deaths in any interval of length t will be a Poisson random variable with mean μt. Hence,

$$P_{ij}(t) = e^{-\mu t}\frac{(\mu t)^{i-j}}{(i-j)!}, \quad 0 < j \le i$$
$$P_{i0}(t) = e^{-\mu t}\sum_{k=i}^{\infty}\frac{(\mu t)^k}{k!}.$$

10. Let $X(t) = 0$ if machine 1 is working at time t and $X(t) = 1$ otherwise. Similarly, let $Y(t) = 0$ if machine 1 is working at time t and $Y(t) = 1$ otherwise. It follows from Example 6.11 that the transition probabilities of (X_t) are given by

$$Q_{00}(t) = \frac{\lambda_1}{\lambda_1+\mu_1}e^{-(\lambda_1+\mu_1)t} + \frac{\mu_1}{\lambda_1+\mu_1}$$
$$Q_{01}(t) = -\frac{\lambda_1}{\lambda_1+\mu_1}e^{-(\lambda_1+\mu_1)t} + \frac{\lambda_1}{\lambda_1+\mu_1}$$
$$Q_{10}(t) = \frac{\mu_1}{\lambda_1+\mu_1} - \frac{\mu_1}{\lambda_1+\mu_1}e^{-(\lambda_1+\mu_1)t}$$
$$Q_{11}(t) = \frac{\lambda_1}{\lambda_1+\mu_1} + \frac{\mu_1}{\lambda_1+\mu_1}e^{-(\lambda_1+\mu_1)t}$$

and the transition probabilities of (Y_t) are given by

$$R_{00}(t) = \frac{\lambda_2}{\lambda_2+\mu_2}e^{-(\lambda_2+\mu_2)t} + \frac{\mu_2}{\lambda_2+\mu_2}$$
$$R_{01}(t) = -\frac{\lambda_2}{\lambda_2+\mu_2}e^{-(\lambda_2+\mu_2)t} + \frac{\lambda_2}{\lambda_2+\mu_2}$$
$$R_{10}(t) = \frac{\mu_2}{\lambda_2+\mu_2} - \frac{\mu_2}{\lambda_2+\mu_2}e^{-(\lambda_2+\mu_2)t}$$
$$R_{11}(t) = \frac{\lambda_2}{\lambda_2+\mu_2} + \frac{\mu_2}{\lambda_2+\mu_2}e^{-(\lambda_2+\mu_2)t}.$$

The process $Z(t) = (X(t), Y(t))$ is a continuous-time Markov chain with state space $\{(0,0),(0,1),(1,0),(1,1)\}$. By independence, the transition probabilities of $(Z(t))$ are given by

$$
\begin{aligned}
P_{(0,0),(0,0)}(t) &= Q_{00}(t)R_{00}(t)\\
P_{(0,0),(0,1)}(t) &= Q_{00}(t)R_{01}(t)\\
P_{(0,0),(1,0)}(t) &= Q_{01}(t)R_{00}(t)\\
P_{(0,0),(1,1)}(t) &= Q_{01}(t)R_{01}(t)\\
P_{(0,1),(0,0)}(t) &= Q_{00}(t)R_{10}(t)\\
P_{(0,1),(0,1)}(t) &= Q_{00}(t)R_{11}(t)\\
P_{(0,1),(1,0)}(t) &= Q_{01}(t)R_{10}(t)\\
P_{(0,1),(1,1)}(t) &= Q_{01}(t)R_{11}(t)\\
P_{(1,0),(0,0)}(t) &= Q_{10}(t)R_{00}(t)\\
P_{(1,0),(0,1)}(t) &= Q_{10}(t)R_{01}(t)\\
P_{(1,0),(1,0)}(t) &= Q_{11}(t)R_{00}(t)\\
P_{(1,0),(1,1)}(t) &= Q_{11}(t)R_{01}(t)\\
P_{(1,1),(0,0)}(t) &= Q_{10}(t)R_{10}(t)\\
P_{(1,1),(0,1)}(t) &= Q_{10}(t)R_{11}(t)\\
P_{(1,1),(1,0)}(t) &= Q_{11}(t)R_{10}(t)\\
P_{(1,1),(1,1)}(t) &= Q_{11}(t)R_{11}(t).
\end{aligned}
$$

It is easy to check that these transition probabilities satisfy the forward and backward equations.

11. See the solution at the end of the book.

13. With the number of customers in the shop as the state, we get a birth and death process with

$$\lambda_0 = \lambda_1 = 3, \mu_1 = \mu_2 = 4$$

Therefore

$$P_1 = \frac{3}{4}P_0, \quad P_2 = \frac{3}{4}P_1 = (\frac{3}{4})^2 P_0.$$

Hence

$$P_0 = \left(1 + \frac{3}{4} + (\frac{3}{4})^2\right)^{-1} = \frac{16}{37}.$$

(a) The average number of customers in the shop is

$$P_1 + 2P_2 = \left(\frac{3}{4} + 2(\frac{3}{4})^2\right) P_0 = \frac{30}{37}.$$

(b) The proportion of customers that enter the shop is

$$\frac{\lambda(1 - P_2)}{\lambda} = 1 - P_2 = \frac{28}{37}.$$

(c) Now $\mu = 8$ and so

$$P_0 = \left(1 + \frac{3}{8} + (\frac{3}{8})^2\right)^{-1} = \frac{64}{97}.$$

So the proportion of customers who now enter the shop is

$$1 - P_2 = \frac{88}{97}.$$

The rate of added customers is therefore

$$\lambda\left(\frac{88}{97} - \frac{28}{37}\right) = 3\left(\frac{88}{97} - \frac{28}{37}\right) = 0.45$$

The business he does would improve by 0.45 customers per hour.

14. Similar to 13.

15. With the number of customers in the system as the state, we get a birth and death process with

$$\lambda_0 = \lambda_1 = \lambda_2 = 3, \lambda_3 = 0$$
$$\mu_0 = 0, \mu_1 = 2, \mu_2 = \mu_3 = 4$$

Therefore, the balance equations reduce to

$$P_1 = \frac{3}{2}P_0, \quad P_2 = \frac{3}{4}P_1 = \frac{9}{8}P_0, \quad P_3 = \frac{3}{4}P_2 = \frac{27}{32}P_0.$$

Therefore,

$$P_0 = \left(1 + \frac{3}{2} + \frac{9}{8} + \frac{27}{32}\right)^{-1} = \frac{32}{143}.$$

(a) The fraction of potential customers that enter the system is

$$\frac{\lambda(1 - P_3)}{\lambda} = 1 - P_3 = 1 - \frac{27}{32}\frac{32}{143} = \frac{116}{143}.$$

(b) With a server working twice as fast we would get

$$P_1 = \frac{3}{4}P_0$$
$$P_2 = \frac{3}{4}P_1 = (\frac{3}{4})^2 P_0$$
$$P_3 = \frac{3}{4}P_2 = (\frac{3}{4})^3 P_0$$

and

$$P_0 = \left(1 + \frac{3}{4} + (\frac{3}{4})^2 + (\frac{3}{4})^3\right)^{-1} = \frac{64}{175}$$

Thus

$$1 - P_3 = \frac{148}{175}.$$

16. See the solution at the end of book.

17. Say the state is 0 if the machine is up, say it is i when it is down due to a type i failure, $i = 1, 2$.The balance equations for the limiting probabilities are as follows.

$$\lambda P_0 = \mu_1 P_1 + \mu_2 P_2$$
$$\mu 1 P_1 = \lambda p P_0$$
$$\mu_2 P_2 = \lambda(1p) P_0$$
$$P_0 + P_1 + P_2 = 1$$

These equations are easily solved to give the results

$$P_0 = (1 + \lambda p/\mu_1 + \lambda(1-p)/\mu_2)^{-1},$$
$$P_1 = \lambda p P_0/\mu_1,$$
$$P_2 = \lambda(1-p) P_0/\mu_2.$$