Sketch of solutions to HW8

Chapter 5

49. (a)
$$P(N(T) - N(s) = 1) = \lambda(T-s)e^{-\lambda(T-s)}.$$

(b) Differentiating the expression in part (a) and then setting it equal to 0 gives
$$\lambda(T-s)e^{-\lambda(T-s)} = e^{-\lambda(T-s)}$$
implying that the maximizing value is $s = T - \frac{1}{\lambda}$.

(c) For $s = T - \frac{1}{\lambda}$, we have $\lambda(T-s) = 1$. Thus
$$P(N(T) - N(s) = 1) = e^{-1}.$$

60. Given $N(1) = 2$, the two arrival times S_1, S_2 have the same distribution as the order statistics corresponding to 2 independent random variables uniformly distributed over the interval $(0,1)$. Thus the conditional joint density of S_1 and S_2 given $N(1) = 2$ is
$$f(s_1, s_2|2) = 2, \qquad 0 < s_1 < s_2 < 1.$$

(a) Thus the probability that both arrived in the first 20 minutes is equal to
$$\int_0^{1/3}\int_0^{s_2} 2ds_1ds_2 = \frac{1}{9}.$$

(b) The conditional marginal density of S_1 given $N(1) = 2$ is
$$f(s_1) = 2(1-s_1), \qquad 0 < s_1 < 1.$$
The probability that at least one arrived during the first 20 minutes is equal to
$$\int_0^{1/3} 2(1-s_1)ds_1 = \frac{5}{9}.$$

78. Let
$$\lambda(t) = \begin{cases} 4 & 0 \le t \le 2, \\ 8 & 2 < t \le 4, \\ 4+t & 4 < t \le 6, \\ 22-2t & 6 < t \le 9. \end{cases}$$
The number of customers that enter the store on a given day is $N(9)$ which is a Poisson random variable with mean
$$m(9) = \int_0^9 \lambda(t)dy = 63.$$

85. The mean is

$$5 \cdot 4 \cdot 2000 = 40000.$$

The variance is

$$5 \cdot 4 \cdot 8000000 = 160000000.$$

87.

$$\begin{aligned}
\mathrm{Cov}[X(t), X(t+s)] &= \mathrm{Cov}[X(t), X(t) + X(t+s) - X(t)] \\
&= \mathrm{Cov}[X(t), X(t)] + \mathrm{Cov}[X(t), X(t+s) - X(t)] \\
&= \mathrm{Var}(X(t)) = \lambda t E[Y_1^2].
\end{aligned}$$

Chapter 6

4. See the solution at the end of the book.

5. (a) Yes.
(b) It is a pure birth process.
(c) If there are i infected individuals then since a contact will involve an infected and an uninfected individual with probability

$$\frac{i(n-i)}{\binom{n}{2}},$$

it follows that the birth rates are

$$\lambda_i = \frac{i(n-i)}{\binom{n}{2}} \lambda p, \quad i = 1, \ldots, n.$$

Hence,

$$E[\text{time until all are affected}] = \frac{n(n-1)}{2\lambda p} \sum_{i=1}^{n-1} \frac{1}{i(n-i)}.$$

6. (a) and (b) We know that

$$\begin{aligned}
E[T_0] &= \frac{1}{\lambda} \\
E[T_1] &= \frac{1}{2\lambda} + \frac{\mu}{2\lambda} E[T_0] = \frac{1}{2\lambda} + \frac{\mu}{2\lambda}\frac{1}{\lambda} = \frac{1}{2\lambda}(1 + \frac{\mu}{\lambda}) \\
E[T_2] &= \frac{1}{3\lambda} + \frac{2\mu}{3\lambda} E[T_1] = \frac{1}{3\lambda}(1 + \frac{\mu}{\lambda} + (\frac{\mu}{\lambda})^2) \\
E[T_3] &= \frac{1}{4\lambda} + \frac{3\mu}{4\lambda} E[T_2] = \frac{1}{4\lambda}(1 + \frac{\mu}{\lambda} + (\frac{\mu}{\lambda})^2 + (\frac{\mu}{\lambda})^3) \\
E[T_4] &= \frac{1}{5\lambda} + \frac{4\mu}{5\lambda} E[T_2] = \frac{1}{5\lambda}(1 + \frac{\mu}{\lambda} + (\frac{\mu}{\lambda})^2 + (\frac{\mu}{\lambda})^3 + (\frac{\mu}{\lambda})^4).
\end{aligned}$$

Thus

$$E[\text{time to go from 0 to 4}] = E[T_0] + E[T_1] + E[T_2] + E[T_3]$$

and

$$E[\text{time to go from 2 to 5}] = E[T_2] + E[T_3] + E[T_4].$$

(c) We know that

$$\begin{aligned}
\text{Var}(T_0) &= \frac{1}{\lambda^2} \\
\text{Var}(T_1) &= \frac{1}{2\lambda(2\lambda+\mu)} + \frac{\mu}{2\lambda}\text{Var}(T_0) + \frac{\mu}{(2\lambda+\mu)}(E[T_0] + E[T_1])^2 \\
&= \frac{1}{2\lambda(2\lambda+\mu)} + \frac{\mu}{2\lambda}\frac{1}{\lambda^2} + \frac{\mu}{(2\lambda+\mu)}\left(\frac{1}{\lambda} + \frac{1}{2\lambda}(1+\frac{\mu}{\lambda})\right)^2 \\
\text{Var}(T_2) &= \frac{1}{3\lambda(3\lambda+2\mu)} + \frac{2\mu}{3\lambda}\text{Var}(T_1) + \frac{2\mu}{(3\lambda+2\mu)}(E[T_1] + E[T_2])^2 \\
&= \frac{1}{3\lambda(3\lambda+2\mu)} + \frac{2\mu}{3\lambda}\left(\frac{1}{2\lambda(2\lambda+\mu)} + \frac{\mu}{2\lambda}\frac{1}{\lambda^2} + \frac{\mu}{(2\lambda+\mu)}\left(\frac{1}{\lambda} + \frac{1}{2\lambda}(1+\frac{\mu}{\lambda})\right)^2\right) \\
&+ \frac{2\mu}{(3\lambda+2\mu)}\left(\frac{1}{2\lambda}(1+\frac{\mu}{\lambda}) + \frac{1}{3\lambda}(1+\frac{\mu}{\lambda}+(\frac{\mu}{\lambda})^2)\right)^2 \\
\text{Var}(T_3) &= \frac{1}{4\lambda(4\lambda+3\mu)} + \frac{3\mu}{4\lambda}\text{Var}(T_2) + \frac{3\mu}{(4\lambda+3\mu)}(E[T_2] + E[T_3])^2 \\
\text{Var}(T_4) &= \frac{1}{5\lambda(5\lambda+4\mu)} + \frac{4\mu}{5\lambda}\text{Var}(T_3) + \frac{4\mu}{(5\lambda+4\mu)}(E[T_3] + E[T_4])^2.
\end{aligned}$$

Thus

$$\text{Var}\,(\text{time to go from 0 to 4}) = \text{Var}(T_0) + \text{Var}(T_1) + \text{Var}(T_2) + \text{Var}(T_3)$$

and

$$\text{Var}\,(\text{time to go from 2 to 5}) = \text{Var}(T_2) + \text{Var}(T_3) + \text{Var}(T_4).$$