Sketch of solutions to HW7

Chapter 5

6. Let A be the event that Mr. Jones finishes before Mr. Brown and let E be the event that is not the last to leave the post office. Then, by conditioning, we have

$$P(E) = P(E|A)P(A) + P(E|A^c)P(A^c) = P(E|A)\frac{\lambda_1}{\lambda_1+\lambda_2} + P(E|A^c)\frac{\lambda_2}{\lambda_1+\lambda_2}.$$

By using the memoryless property of exponential random variables, we get

$$P(E|A) = \frac{\lambda_1}{\lambda_1+\lambda_2}, \qquad P(E|A^c) = \frac{\lambda_2}{\lambda_1+\lambda_2}.$$

Thus

$$P(E) = \left(\frac{\lambda_1}{\lambda_1+\lambda_2}\right)^2 + \left(\frac{\lambda_2}{\lambda_1+\lambda_2}\right)^2.$$

7. See the solution at the end of the book

8. For any $x > 0$,

$$P(X > x|X < Y) = \frac{P(X > x, X < Y)}{P(X < Y)} = \frac{P(x < X < Y)}{P(X < Y)}.$$

We know that

$$P(X < Y) = \frac{\lambda}{\lambda+\mu}$$

and

$$\begin{aligned} P(x < X < Y) &= \int_x^\infty P(x < X < Y|X = t)\lambda e^{-\lambda t}dt \\ &= \int_x^\infty P(t < Y)\lambda e^{-\lambda t}dt = \int_x^\infty \lambda e^{-(\lambda+\mu)t}dt \\ &= \frac{\lambda}{\lambda+\mu}e^{-(\lambda+\mu)x}. \end{aligned}$$

Thus

$$P(X > x|X < Y) = e^{-(\lambda+\mu)x}.$$

That is, given $X < Y$, X is an exponential random variable with parameter $\lambda + \mu$.

9. Let A be the even that machine 1 is still working at time t, and let E be the event that machine 1 is the first to fail. Then, by conditioning,

$$P(E) = P(E|A)P(A) + P(E|A^c)P(A^c) = P(E|A)e^{-\lambda_1 t} + P(E|A^c)(1 - e^{-\lambda_1 t}).$$

Note that $P(E|A^c) = 1$ and, by the memeoryless property of exponential random variables,

$$P(E|A) = \frac{\lambda_1}{\lambda_1+\lambda_2}.$$

Thus

$$P(E) = \frac{\lambda_1}{\lambda_1 + \lambda_2} e^{-\lambda_1 t} + (1 - e^{-\lambda_1 t}).$$

10. See the solution at the end of the book.

15. For $i = 1, 2, \ldots, 5$, let T_i be the time between the $(i-1)$-th and the i-th failures. Then by the memoryless property of exponential random variables, $T_1, T_2, \ldots, T_5$ are independent and, T_i is an exponential random variable with rate $(101 - i)/200$, $i = 1, 2, \ldots, t$. Thus

$$E[T] = E[\sum_{i-1}^{5} T_i = \sum_{i=1}^{5} E[T_i] = \sum_{i=1}^{5} \frac{200}{101 - i}$$

and

$$\text{Var}(T) = \text{Var}(\sum_{i-1}^{5} T_i) = \sum_{i=1}^{5} \text{Var}(T_1) \sum_{i=1}^{5} \left(\frac{200}{101 - i}\right)^2.$$

18. See the solution at the end of the book.

19. See the solution at the end of the book.

36. See the solution at the end of the book.

38. (a)

$$P(N_1(t) = n, N_2(t) = m) = \sum_{k=0}^{\min(n,m)} P(N_1(t) = n, N_2(t) = m, M_2(t) = k)$$

$$= \sum_{k=0}^{\min(n,m)} P(M_1(t) = n - k, M_2(t) = k, M_3(t) = m - k)$$

$$= \sum_{k=0}^{\min(n,m)} P(M_1(t) = n - k)P(M_2(t) = k)P(M_3(t) = m - k)$$

$$= \sum_{k=0}^{\min(n,m)} \frac{\lambda_1^{n-k}}{(n-k)!} e^{-\lambda_1 t} \frac{\lambda_2^k}{k!} e^{-\lambda_2 t} \frac{\lambda_3^{m-k}}{(m-k)!} e^{-\lambda_3 t}$$

$$= e^{-(\lambda_1 + \lambda_2 + \lambda_3)t} \sum_{k=0}^{\min(n,m)} \frac{\lambda_1^{n-k}}{(n-k)!} \frac{\lambda_2^k}{k!} \frac{\lambda_3^{m-k}}{(m-k)!}.$$

(b)

$$\text{Cov}(N_1(t), N_2(t)) = \text{Cov}(M_1(t) + M_2(t), M_2(t) + M_3(t))$$
$$= \text{Cov}(M_1(t), M_2(t)) + \text{Cov}(M_1(t), M_3(t)) + \text{Cov}(M_2(t), M_2(t)) + \text{Cov}(M_2(t), M_3(t))$$
$$= \text{Cov}(M_2(t), M_2(t)) = \text{Var}(M_2(t)) = \lambda_2 t.$$

42. (a) $E[S_4] = 4/\lambda$.

(b) Given $N(1) = 2$, the conditional distribution of S_4 is the same as the unconditional distribution of $1 + S_2$. Thus

$$E[S_4|N(1) = 2] = 1 + \frac{2}{\lambda}.$$

(c) By using the independent increment property and the stationary increments property,

$$E[N(4) - N(2)|N(1) = 3] = E[N(4) - N(2)] = E[N(2)] = 2\lambda.$$

45. (a) Note that for $n = 0, 1, \ldots,$

$$E[N(T)|T = n] = E[N(n)|T = n] = E[N(n)] = n\lambda,$$

we have $E[N(T)|T] = T\lambda$. Thus $E[N(T)] = E[E[N(T)|T]] = \mu\lambda$. We have $E[TN(T)|T] = TE[N(T)|T] = T^2\lambda$. Thus

$$E[TN(T)] = E[E[TN(T)|T]] = E[T^2]\lambda = (\sigma^2 + \mu^2)\lambda.$$

Therefore

$$\mathrm{Cov}(T, N(T)) = E[TN(T)] - E[T]E[N(T)] = (\sigma^2 + \mu^2)\lambda - \mu^2\lambda = \sigma^2\lambda.$$

(b) Note that for $n = 0, 1, \ldots,$

$$E[N(T)^2|T = n] = E[N(n)^2|T = n] = E[N(n)^2] = n\lambda + n^2\lambda^2,$$

we have

$$E[N(T)^2|T] = T\lambda + T^2\lambda^2.$$

Thus

$$\begin{aligned} E[N(T)^2] &= E[E[N(T)^2|T]] = E[T\lambda + T^2\lambda^2] \\ &= \mu\lambda + (\sigma^2 + \mu^2)\lambda^2. \end{aligned}$$

Therefore

$$\mathrm{Var}(N(T)) = \mu\lambda + (\sigma^2 + \mu^2)\lambda^2 - (\mu\lambda)^2 = \mu\lambda + \sigma^2\lambda^2.$$