Sketch of solutions to HW6

Chapter 4

64 We know that if \( X_0 = 1 \), then \( E[X_n] = \mu^n \). Thus the expected number of individuals that ever exist in this population is
\[
\sum_{n=0}^{\infty} \mu^n = \frac{1}{1 - \mu}.
\]

If \( X_0 = n \), by using independence, we see that in this case the expected number of individuals that ever exist in this population is \( n/(1 - \mu) \).

65. Recall that \( \pi_0 = \lim_{n \to \infty} P(X_n = 0|X_0 = 1) \).

Suppose that \( \pi \) is a positive solution to the equation
\[
\pi = \sum_{j=0}^{\infty} \pi^j P_j,
\]
then \( \pi > 0 = P(X_0 = 0|X_0 = 1) \). Assume that \( \pi \geq P(X_{n-1} = 0|X_0 = 1) \). Then
\[
P(X_n = 0|X_0 = 1) = \sum_{j=0}^{\infty} P(X_n = 0|X_1 = j)P_j
\]
\[
= \sum_{j=0}^{\infty} P(X_n = 0|X_1 = 1)^j P_j
\]
\[
= \sum_{j=0}^{\infty} P(X_{n-1} = 0|X_0 = 1)^j P_j
\]
\[
\leq \sum_{j=0}^{\infty} \pi^j P_j = \pi.
\]
So we have proved by induction that \( \pi \geq P(X_n = 0|X_0 = 1) \) for all \( n \geq 0 \). Thus \( \pi \geq \pi_0 \).

That is, \( \pi_0 \) is the smallest positive solution of
\[
\pi = \sum_{j=0}^{\infty} \pi^j P_j.
\]

66. (a) Solving the equation
\[
\pi + \frac{1}{4} + \pi^2 \frac{3}{4},
\]
we get \( \pi = \frac{1}{3} \) or \( \pi = 1 \). Thus \( \pi_0 = \frac{1}{3} \).

(b) In this case the mean number of offspring \( \mu = 1 \), so \( \pi_0 = 1 \).

(c) Solving the equation
\[
\pi = \frac{1}{6} + \pi \frac{1}{2} + \pi^3 \frac{1}{3},
\]
we get only two positive solutions which are \((\sqrt{2} - 1)/2\) and 1. So \(\pi_0 = (\sqrt{3} - 1)/2\).

70. (a) \(P_{0,1} = P_{m,m-1} = 1\). For \(i = 1, \ldots, m - 1\),

\[
P_{i,i} = \frac{2i(m-i)}{m^2} \quad P_{i,i+1} = \frac{(m-i)^2}{m^2} \quad P_{i,i-1} = \frac{i^2}{m^2}.
\]

(c) From the system

\[
\begin{align*}
\pi_0 &= \pi_1 \frac{1}{m^2} \\
\pi_1 &= \pi_0 + \pi_1 \frac{2(m-1)}{m^2} + \pi_2 \frac{2^2}{m^2} \\
\pi_2 &= \pi_1 \frac{(m-1)^2}{m^2} + \pi_2 \frac{2 \cdot 2 \cdot (m-2)}{m^2} + \pi_3 \frac{3^2}{m^3} \\
&\vdots \\
\pi_m &= \pi_{m-1} \frac{1}{m^2} \\
\pi_0 + \pi_1 + \cdots + \pi_m &= 1
\end{align*}
\]

we get

\[
\begin{align*}
\pi_1 &= \pi_0 m^2 = \pi_0 \left( \frac{m}{1} \right)^2 \\
\pi_2 &= \pi_1 \frac{(m-1)^2}{2^2} = \pi_0 m^2 \frac{(m-1)^2}{2^2} = \pi_0 \left( \frac{m}{2} \right)^2 \\
\pi_3 &= \pi_2 \frac{(m-2)^2}{3^2} = \pi_0 m^2 \frac{(m-1)^2}{2^2} \frac{(m-2)^2}{3^2} = \pi_0 \left( \frac{m}{3} \right)^2 \\
&\vdots \\
\pi_i &= \pi_0 m^2 \frac{(m-1)^2}{2^2} \frac{(m-2)^2}{3^2} \cdots \frac{(m-i+1)^2}{i^2} = \pi_0 \left( \frac{m}{i} \right)^2 \\
&\vdots \\
\pi_m &= \pi_0 = \pi_0 \left( \frac{m}{m} \right)^2
\end{align*}
\]

and

\[
1 = \pi_0 \sum_{i=0}^{m} \left( \frac{m}{i} \right)^2 = \pi_0 \left( \frac{2m}{m} \right).
\]

Thus

\[
\pi_0 = \frac{1}{\left( \frac{2m}{m} \right)}
\]
and
\[ \pi_i = \left( \frac{m_i}{2m} \right)^2 \quad i = 1, \cdots, m. \]

73. It is straightforward to check that \( \pi_i P_{ij} = \pi_j P_{ji} \). For instance \( \pi_0 P_{01} = \frac{1}{5} \frac{1}{2} = \frac{1}{10} \) and \( \pi_1 P_{10} = \frac{2}{5} \frac{1}{4} = \frac{1}{10} \).

Chapter 5

1. (a) \( P(T > \frac{1}{2}) = e^{-1} \).
   (b) By using the memoryless property,
   \[ P(T > 12 | T > 12) = P(T > \frac{1}{2}) = e^{-1}. \]

2. Let \( X_1 \) be the remaining service time of the customer who is being served. Let \( X_i, i = 2, 3, 4, 5 \), be the service times of the customers waiting in line. Let \( X_6 \) be your service time. Then \( X_1, \ldots, X_6 \) are exponential random variables with parameter \( \mu \). The time you spend in the bank is \( X_1 + \cdots + X_6 \). So your expected amount of time spent in the bank is \( 6/\mu \).

3. By using the memoryless property, (a) is the only correct answer.

4. (a) 0.
   (b) Let \( E \) be the event that \( A \) is still in the post office after the other two have left. The only scenario for this to happen is when the service time of \( B \) is 1, the service time of \( C \) is 1 and the service time of \( A \) is 3. So the answer is 3^{-3}.
   (c) Let \( X_1 \) be the service time of \( A \), \( X_2 \) the service of \( B \), \( X_3 \) the service time of \( C \). Then \( X_1, X_2 \) and \( X_3 \) are independent exponential random variables with parameter \( \mu \), and \( X_2 + X_3 \) is a gamma random variable with parameters \( (2, \mu) \). Thus
   \[ P(X_1 > X_2 + X_3) = \int_0^\infty P(X_1 > x)P(X_2 + X_3 = x)(\mu x)e^{-\mu x}dx \]
   \[ = \int_0^\infty P(X_1 > x)(\mu x)e^{-\mu x}dx = \int_0^\infty P(X_1 > x)(\mu x)e^{-2\mu x}dx = \frac{1}{4}. \]