Sketch of solutions to HW5

Chapter 4

19. Solving the system

$$\begin{aligned}
\pi_0 &= 0.7\pi_0 + 0.5\pi_1 \\
\pi_1 &= 0.4\pi_2 + 0.2\pi_3 \\
\pi_2 &= 0.3\pi_0 + 0.5\pi_1 \\
\pi_3 &= 0.6\pi_2 + 0.8\pi_3 \\
\pi_0 &+ \pi_1 + \pi_2 + \pi_3 = 1,
\end{aligned}$$

we get $\pi_0 = \frac{1}{4}$, $\pi_1 = \frac{3}{20}$, $\pi_2 = \frac{3}{20}$ and $\pi_3 = \frac{9}{20}$.

20. Using the fact that $\mathbf{P}$ is doubly stochastic, that is

$$\sum_i P_{ij} = 1, \quad \text{for all } j,$$

we can easily see that $\pi_j = \frac{1}{M+1}$ satisfies

$$\pi_j = \sum_i \pi_i P_{ij}, \quad \text{for all } j.$$

We clearly have $\sum_j \pi_j = 1$, so $\pi_0, \pi_1, \ldots, \pi_m$ are the long-rum proportions.

22. Let X_n be the remainder when Y_n is divided by 13, that is, we let $X_n = i$, $i = 0, 1, \ldots, 12$, if $Y_n = 13k+i$ for some integer k. Then X_n is a Markov chain on the state space $\{0, 1, \ldots, 12\}$ with transition matrix

$$P = \begin{pmatrix}
0 & \frac16 & \frac16 & \frac16 & \frac16 & \frac16 & \frac16 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac16 & \frac16 & \frac16 & \frac16 & \frac16 & \frac16 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \frac16 & \frac16 & \frac16 & \frac16 & \frac16 & \frac16 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \frac16 & \frac16 & \frac16 & \frac16 & \frac16 & \frac16 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \frac16 & \frac16 & \frac16 & \frac16 & \frac16 & \frac16 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \frac16 & \frac16 & \frac16 & \frac16 & \frac16 & \frac16 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac16 & \frac16 & \frac16 & \frac16 & \frac16 & \frac16 \\
\frac16 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac16 & \frac16 & \frac16 & \frac16 & \frac16 \\
\frac16 & \frac16 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac16 & \frac16 & \frac16 & \frac16 \\
\frac16 & \frac16 & \frac16 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac16 & \frac16 & \frac16 \\
\frac16 & \frac16 & \frac16 & \frac16 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac16 & \frac16 \\
\frac16 & \frac16 & \frac16 & \frac16 & \frac16 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac16 \\
\frac16 & \frac16 & \frac16 & \frac16 & \frac16 & \frac16 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}.$$

This transition matrix is doubly stochastic and

$$\lim_{n\to\infty} P(Y_n \text{ is a multiple of } 13) = \frac{1}{13}.$$

23. (a) Letting 0 stand for a good year and 1 for a bad year, the successive states follow a Markov chain with transition probability matrix

$$P = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{3} & \frac{2}{3} \end{pmatrix}.$$

Note that

$$P^2 = \begin{pmatrix} \frac{5}{12} & \frac{7}{12} \\ \frac{7}{18} & \frac{11}{18} \end{pmatrix}.$$

Hence if S_i is the number of storms in year i, then

$$E[S_1] = E[S_1|X_1 = 0]P_{00} + E[S_1|X_1 = 1]P_{01} = \frac{1}{2} + \frac{3}{2} = 2$$
$$E[S_2] == E[S_2|X_2 = 0]P_{00}^2 + E[S_1|X_2 = 1]P_{01}^2 = \frac{5}{12} + \frac{21}{12} = \frac{26}{12}.$$

Therefore $E[S_1 + S_2] = \frac{25}{6}$.

(b) By calculating P^3, we can find that $P_{00}^3 = \frac{29}{72}$. Hence

$$P(S_3 = 0) = P(S_3 = 0|X_3 = 0)\frac{29}{72} + P(S_3 = 0|X_3 = 1)\frac{43}{72} = \frac{29}{72}e^{-1} + \frac{43}{72}e^{-3}.$$

(c) By solving the system

$$\pi_0 = \frac{1}{2}\pi_0 + \frac{1}{3}\pi_1$$
$$\pi_1 = \frac{1}{2}\pi_0 + \frac{2}{3}\pi_1$$
$$\pi_0 + \pi_1 = 1,$$

we get $\pi_0 = \frac{2}{5}$, $\pi_1 = \frac{3}{5}$. Thus the long-run average number of storms is $\frac{2}{5} + 3\frac{3}{5} = \frac{11}{5}$.

25. Let X_n denote the number of pairs of shoes at the door the runner departs from at the beginning of day n. Then X_n is a Markov chain with transition probabilities

$$P_{i,i} = \frac{1}{4} \quad P_{i,i-1} = \frac{1}{4} \quad P_{i,k-i} = \frac{1}{4} \quad P_{i,k-i+1} = \frac{1}{4} \quad 0 < i < k.$$

and

$$P_{0,0} = \frac{1}{2} \quad P_{0,k} = \frac{1}{2} \quad P_{k,k} = \frac{1}{4} \quad P_{k,0} = \frac{1}{4} \quad P_{k,1} = \frac{1}{4} \quad P_{k,k-1} = \frac{1}{4}.$$

The transition matrix is doubly stochastic, so the propor- tion of time the runner runs barefooted is $\frac{1}{k+1}$.

31. Let the state on day n be 0 if sunny, 1 if cloudy,and 2 if rainy. This gives a three-state Markov chain with transition probability matrix

$$P = \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \end{pmatrix}.$$

Using this, one can easily get $\pi_0 = \frac{1}{5}, \pi_1 = \frac{2}{5}, \pi_2 = \frac{2}{5}$.

33. Consider the Markov chain whose state at time n is the type of exam number n. The transition probabilities of this Markov chain are obtained by conditioning on the performance of the class. This gives the following:

$$\begin{aligned} P_{11} &= 0.3 \cdot \frac{1}{3} + 0.7 \cdot 1 = 0.8, \quad P_{12} = P_{13} = 0.3 \cdot \frac{1}{3} = 0.1 \\ P_{21} &= 0.6 \cdot \frac{1}{3} + 0.4 \cdot 1 = 0.6, \quad P_{22} = P_{23} = 0.6 \cdot \frac{1}{3} = 0.2 \\ P_{31} &= 0.9 \cdot \frac{1}{3} + 0.1 \cdot 1 = 0.4 \quad P_{32} = P_{33} = 0.9 \cdot \frac{1}{3} = 0.3 \end{aligned}$$

Using this, one can easily find that $\pi_1 = \frac{5}{7}, \pi_2 = \frac{1}{7}, \pi_3 = \frac{1}{7}$.

35. By solving the system

$$\begin{aligned} \pi_0 &= \pi_1 + \frac{1}{2}\pi_2 + \frac{1}{3}\pi_3 + \frac{1}{4}\pi_4 \\ \pi_1 &= \frac{1}{2}\pi_2 + \frac{1}{3}\pi_3 + \frac{1}{4}\pi_4 \\ \pi_2 &= \frac{1}{3}\pi_3 + \frac{1}{4}\pi_4 \\ \pi_3 &= \frac{1}{4}\pi_4 \\ \pi_4 &= \pi_0 \\ \pi_0 &+ \pi_1 + \pi_2 + \pi_3 + \pi_4 = 1 \end{aligned}$$

we get that $\pi_0 = \pi_4 = \frac{12}{37}$, $\pi_1 = \frac{6}{37}$, $\pi_2 = \frac{4}{37}$ $\pi_3 = \frac{3}{37}$.

56. This is equivalent to the gambler's ruin with $N = n + m$ starting with i units. So the desired answer is

$$\frac{1 - (q/p)^m}{1 - (q/p)^{m+n}}$$

when $p \neq \frac{1}{2}$ (where $q = 1 - p$), and the desired answer is $m/(n + m)$ when $p = \frac{1}{2}$.

57. Let A be the event that all states have been visited by time T. Then, conditioning on the direction of the first step gives

$$\begin{aligned} P(A) &= P(A|\text{clockwise})p + P(A|\text{counterclockwise})q \\ &= p\frac{1 - (q/p)}{1 - (q/p)^n} + q\frac{1 - (p/q)}{1 - (p/q)^n}. \end{aligned}$$

The conditional probabilities in the preceding follow by noting that they are equal to the probability in the gambler's ruin problem that a gambler that starts with 1 will reach n before going broke when the gambler's win probabilities are p and q respectively.