Sketch of solutions to HW4

Chapter 4

1.

$$P = \begin{pmatrix} 0 & 1 & 0 & 0 \\ \frac{1}{9} & \frac{4}{9} & \frac{4}{9} & 0 \\ 0 & \frac{4}{9} & \frac{4}{9} & \frac{1}{9} \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

4. See the end of book for the solution.

5.

$$P^3 = \begin{pmatrix} \frac{13}{36} & \frac{11}{54} & \frac{47}{108} \\ \frac{4}{9} & \frac{4}{27} & \frac{11}{27} \\ \frac{5}{12} & \frac{2}{9} & \frac{13}{36} \end{pmatrix}.$$

Thus

$$\begin{aligned} E[X_3] &= P(X_3 = 1) + 2P(X_3 = 2) \\ &= \frac{1}{4}P^3_{01} + \frac{1}{4}P^3_{11} + \frac{1}{2}P^3_{21} + 2\left(\frac{1}{4}P^3_{02} + \frac{1}{4}P^3_{12} + \frac{1}{2}P^3_{22}\right). \end{aligned}$$

7. The desired probability is

$$P^2_{30} + P^2_{31} = 0.26.$$

10. Like on page 191, we define

$$N = \min\{n : X_n = 2\}$$

and construct a Markov chain W with $\mathcal{A} = \{2\}$. Then the transition matrix of this Markov chain is

$$Q = \begin{pmatrix} .5 & .4 & .1 \\ .3 & .4 & .3 \\ 0 & 0 & 1 \end{pmatrix}.$$

We are looking for $P(N > 3|X_0 = 0) = 1 - P(N \leq 3|X_0 = 0)$. As in Example 4.12, $P(N \leq 3|X_0 = 0) = Q^3_{0,2}$ which can be calculated easily, from which one can get the desired answer.

14. For P_1, there is only one class, $\{0, 1, 2\}$. It is recurrent.

For P_2, there is only one class, $\{0, 1, 2, 3\}$. It is recurrent.

For P_3, the classes are $\{0, 2\}$, $\{1\}$ and $\{3, 4\}$. The class $\{1\}$ is transient, the other two classes are recurrent.

For P_4, the classes are $\{0, 1\}$, $\{2\}$, $\{3\}$ and$\{4\}$. The classes $\{3\}$ and$\{4\}$ are transient, and the classes $\{0, 1\}$ and $\{2\}$ are recurrent.

15. If j is reachable from i, then there is a path from i to j that will has positive probability. Taking away all the loops from this path, we get a path of length at most M that still has positive probability.

16. See the end of book for the proof.