Sketch of solutions to HW3

Chapter 3

24. (a) Let X be the number of flips needed. Let $Y = 1$ if the first flip is a head and $Y = 0$ if the first flip is a tail. Given that the first flip is a head, the number of flips needed is equal to 1 plus a geometric random variable with parameter $1-p$. Given that the first flip is a tail, the number of flips needed is equal to 1 plus a geometric random variable with parameter p. Thus

$$E[X] = P(Y=1)E[X|Y=1] + P(Y=0)E[X|Y=0] = p\left(1+\frac{1}{1-p}\right) + (1-p)\left(1+\frac{1}{p}\right).$$

25. (a) For $i = 1, 2, 3$, Let $Y = i$ if the first trial ends up with out come i. Then

$$\begin{aligned} E[N] &= P(Y=1)E[N|Y=1] + P(Y=2)E[N|Y=2] + P(Y=3)E[N|Y=3] \\ &= p_1E[N|Y=1] + p_2E[N|Y=2] + p_3E[N|Y=3]. \end{aligned}$$

Given the first trial ends up with out come i, N is equal to 1 plus a negative binomial random variable with parameters $(2, p_i)$ (that is, sums of 2 independent geometric p_i random variables). Thus

$$E[N] = p_1(1+\frac{2}{p_1}) + p_2(1+\frac{2}{p_2}) + p_3(1+\frac{2}{p_3}) = 7.$$

37. Let $Y = 1$ if A is chosen, $Y = 2$ if B is chosen and $Y = 3$ if C is chosen. Then (a)

$$\begin{aligned} E[X] &= \frac{1}{3}\left(E[X|Y=1] + E[X|Y=2] + E[X|Y=3]\right) \\ &= \frac{1}{3}(2.6 + 3 + 3.4) = 3. \end{aligned}$$

(b)

$$\begin{aligned} E[X^2] &= \frac{1}{3}\left(E[X^2|Y=1] + E[X^2|Y=2] + E[X^2|Y=3]\right) \\ &= \frac{1}{3}(2.6 + 2.6^2 + 3 + 3^2 + 3.4 + 3.4^2) \\ &= 3 + \frac{1}{3}(2.6^2 + 3^2 + 3.4^2) \end{aligned}$$

and so

$$\mathrm{Var}(X) = E[X^2] - (E[X])^2 = \frac{1}{3}(2.6^2 + 3^2 + 3.4^2) - 9.$$

38. We are given

$$f_Y(y) = \begin{cases} 1 & y \in (0,1), \\ 0 & \text{otherwise} \end{cases},$$

and

$$f_{X|Y}(x|y) = \begin{cases} \frac{1}{y} & x \in (0, y), y \in (0, 1), \\ 0 & \text{otherwise} \end{cases}.$$

Hence

$$f(x, y) = \begin{cases} \frac{1}{y} & x \in (0, y), y \in (0, 1), \\ 0 & \text{otherwise} \end{cases}.$$

Consequently for $x \in (0, 1)$,

$$f_X(x) = \int_x^1 \frac{1}{y} dy = -\ln x$$

and $f_X(x) = 0$ for $x \notin (0, 1)$. So

$$E[X] = -\int_0^1 x \ln x dx = -\frac{y^2}{4}(2\ln y - 1)\,|_x^1 = \frac{1}{4}$$

$$E[X^2] = -\int_0^1 x^2 \ln x dx = -\frac{y^3}{9}(3\ln y - 1)\,|_x^1 = \frac{1}{9}$$

and

$$\text{Var}(X) = E[X^2] - (E[X])^2 = \frac{1}{9} - \frac{1}{16}.$$

44. Let N be the number of customers entering the store on a given day. For each $i = 1, 2, \ldots,$ let X_i be the amount of money spent by customer number i. The problem is implicitly assuming that $N, X_1, X_2, \ldots$ are all independent. We are asked to find $E[\sum_{i=1}^N X_i]$. By the argument in Example 3.10 (or going over the argument again), we have

$$E[\sum_{i=1}^N X_i] = E[N]E[X_1] = 10 \cdot 50 = 500.$$

49. Let Y be the number of wins for A in the first 2 games.

(a) Let E be the even that A is the overall winner. Then

$$\begin{aligned} P(E) &= P(E|Y = 0)P(Y = 0) + P(E|Y = 1)P(Y = 1) + P(E|Y = 2)P(Y = 2) \\ &= 0 + P(E)2p(1 - p) + p^2. \end{aligned}$$

Thus

$$P(E) = \frac{p^2}{1 - 2p(1 - p)}.$$

(b) Let X be the number of games played. Then

$$\begin{aligned} E[X] &= E[X|Y = 0]P(Y = 0) + E[X|Y = 1]P(Y = 1) + E[X|Y = 2]P(Y = 2) \\ &= 2(1 - p)^2 + (2 + E[X])2p(1 - p) + 2p^2. \end{aligned}$$

Thus

$$E[X] = \frac{2}{1-2p(1-p)}.$$

57. Let X be the number of storms in the upcoming rainy season. We are given

$$p_{X|\Lambda}(x|\lambda) = \begin{cases} e^{-\lambda}\frac{\lambda^x}{x!}, & x = 0,1,\ldots,\lambda \in (0,5), \\ 0, & \text{otherwise} \end{cases}$$

and

$$f_\Lambda(\lambda) = \begin{cases} \frac{1}{5}, & \lambda \in (0,5), \\ 0 & \text{otherwise} \end{cases}.$$

Thus

$$\begin{aligned} P(X \geq 3) &= \frac{1}{5}\int_0^5 P(X \geq 3|\Lambda = \lambda)d\lambda = \frac{1}{5}\int_0^5 (1-(1+\lambda+\frac{\lambda^2}{2})e^{-\lambda}d\lambda \\ &= 1 - \frac{1}{5}\int_0^5 (1+\lambda+\frac{\lambda^2}{2})e^{-\lambda}d\lambda. \end{aligned}$$

58. (a)

$$\begin{aligned} E[N] &= \int_0^\infty E[N|Y=y)f_Y(y)dy = \int_0^\infty y\frac{\lambda e^{-\lambda y}(\lambda y)^{r-1}}{(r-1)!}dy \\ &= \frac{r}{\lambda}\int_0^\infty e^{-t}\frac{t^r}{r!}dt = \frac{r}{\lambda}. \end{aligned}$$

(b)

$$\begin{aligned} E[N^2] &= \int_0^\infty E[N^2|Y=y)f_Y(y)dy = \int_0^\infty (y+y^2)\frac{\lambda e^{-\lambda y}(\lambda y)^{r-1}}{(r-1)!}dy \\ &= \frac{r}{\lambda} + \int_0^\infty y^2\frac{\lambda e^{-\lambda y}(\lambda y)^{r-1}}{(r-1)!}dy = \frac{r}{\lambda} + \frac{r(r+1)}{\lambda}\int_0^\infty e^{-\lambda y}\frac{(\lambda y)^{r+1}}{(r+1)!}dy \\ &= \frac{r}{\lambda} + \frac{r(r+1)}{\lambda^2}. \end{aligned}$$

Thus

$$\text{Var}(N) = \frac{r}{\lambda} + \frac{r(r+1)}{\lambda^2} - \frac{r^2}{\lambda^2} = \frac{r}{\lambda} + \frac{r}{\lambda^2}.$$

(c)

$$\begin{aligned} P(N=n) &= \int_0^\infty P(N=n|Y=y)f_Y(y)dy = \int_0^\infty e^{-y}\frac{y^n}{n!}\frac{\lambda e^{-\lambda y}(\lambda y)^{r-1}}{(r-1)!}dy \\ &= \frac{(n+r-1)!}{n!(r-1)!}\lambda^r\int_0^\infty e^{-(\lambda+1)y}\frac{y^{n+r-1}}{(n+r-1)!}dy \\ &= \frac{(n+r-1)!}{n!(r-1)!}\frac{\lambda^r}{(\lambda+1)^{n+r}}\int_0^\infty e^{-t}\frac{t^{n+r-1}}{(n+r-1)!}dt \\ &= \frac{(n+r-1)!}{n!(r-1)!}\frac{\lambda^r}{(\lambda+1)^{n+r}}. \end{aligned}$$

63. (a) Let E be the event that there is exactly one type i coupon in the final collection. Let T be the number of types collected before the first type i coupon appears. Then $P(T = 0) = P(T = 1) = \cdots = P(T = n-1) = \frac{1}{n}$, and for each $i = 0, 1, \ldots, n-1$,

$$P(E|T = i) = \frac{1}{n-j}.$$

The last formula above is because there are still $n - i - 1$ types not collected yet when the first type i is obtained and the probability starting from that point that i is the last of the $n-i$ types (consisting of type i and the uncollected $n-i-1$ types) to be obtained is $1/(n-i)$. Consequently

$$P(E) = \frac{1}{n}\sum_{i=0}^{n-1}\frac{1}{n-i}.$$

74. For $i = 1, 2, 3, 4, 5$, let E_i be the event that the i-component is working and let E be the event that the system is working. Define a random variable Y by let $Y = 1$ if the 3rd component is working and $Y = 0$ otherwise. Then

$$P(E) = p_3 P(E|Y = 1) + (1 - p_3)P(E|Y = 0).$$

$$\begin{aligned} P(E|Y = 1) &= P((E_1 \cup E_2) \cap (E_4 \cup E_5)) = P(E_1 \cup E_2)P(E_4 \cup E_5) \\ &= (p_1 + p_2 - p_1 p_2)(p_4 + p_5 - p_4 p_5) \end{aligned}$$

and

$$\begin{aligned} P(E|Y = 0) &= P((E_1 \cap E_4) \cup (E_2 \cap E_5)) = P(E_1 \cap E_4) + P(E_2 \cap E_5) - P(E_1 \cap E_4 \cap E_2 \cap E_5) \\ &= p_1 p_4 + p_2 p_5 - p_1 p_2 p_4 p_5. \end{aligned}$$

Thus

$$\begin{aligned} P(E) &= p_3(p_1 + p_2 - p_1 p_2)(p_4 + p_5 - p_4 p_5) \\ &\quad + (1 - p_3)(p_1 p_4 + p_2 p_5 - p_1 p_2 p_4 p_5). \end{aligned}$$

92. (a) Let N be the number of coins that Josh spots. We are given that N is a Poisson random variable with parameter 6. Let X be the amount of money in cents that Josh picks up, let A be the number of nickels that Josh picks up, let A be the number of dimes that Josh picks up and let A be the number of quarters that Josh picks up. Then $X = 5A + 10B + 25C$. Thus

$$E[X] = E[5A + 10B + 25C] = 5E[A] + 10E[B] + 25E[C].$$

$$\begin{aligned} E[A] &= \sum_{n=0}^{\infty} E[A|N = n]P(N = n) = \sum_{n=0}^{\infty}\frac{n}{4}P(N = n) \\ &= \frac{1}{4}\sum_{n=0}^{\infty} nP(N = n) = \frac{1}{4}E[N] = \frac{3}{2}. \end{aligned}$$

Similarly, we also have $E[B] = E[C] = \frac{3}{2}$. Thus

$$E[X] = 40 \cdot \frac{3}{2} = 60.$$

94. (a) Regard b as fixed and let $HG(w,r)$ with parameters w and r.

$$\begin{aligned} P(M-1=n) = P(M=n+1) &= \frac{(n+1)P(N=n+1)}{E[N]} \\ &= \frac{(n+1)\binom{w}{n+1}\binom{b}{r-n-1}}{\binom{w+b}{r}} \frac{w+b}{wr} \\ &= \frac{\binom{w-1}{n}\binom{b}{(r-1)-n}}{\binom{(w-1)+b}{r-1}}. \end{aligned}$$

Thus $M-1$ is an $HG(w-1, r-1)$ random variable.

(b) Using (a) and Corollary 3.6, we get the following recursive equation:

$$P_{w,r}(k) = \frac{1}{k}\frac{rw}{w+b}\sum_{j=1}^{k} j\alpha_j P_{w-1,r-1}, k-j).$$

(c)

$$\begin{aligned} P_{w,r}(0) =& P(N=0) = \frac{\binom{b}{r}}{\binom{w+b}{r}} \\ P_{w,r}(1) =& \frac{rw}{w+b}\alpha_1 P_{w-1,r-1}(0) = \frac{rw}{w+b}\frac{\binom{b}{r-1}}{\binom{w-1+b}{r-1}}\alpha_1 \\ P_{w,r}(2) =& \frac{1}{2}\frac{rw}{w+b}\left(\alpha_1 P_{w-1,r-1}(1) + 2\alpha_2 P_{w-1,r-1}(0)\right) \\ =& \frac{1}{2}\frac{rw}{w+b}\frac{(r-1)(w-1)}{w-1+b}\frac{\binom{b}{r-2}}{\binom{w-2+b}{r-1}}\alpha_1 \\ &+ \frac{rw}{w+b}\frac{\binom{b}{r-1}}{\binom{w-1+b}{r-1}}\alpha_2. \end{aligned}$$