Sketch of solutions to HW3

Chapter 3

24. (a) Let $X$ be the number of flips needed. Let $Y = 1$ if the first flip is a head and $Y = 0$ if the first flip is a tail. Given that the first flip is a head, the number of flips needed is equal to 1 plus a geometric random variable with parameter $1 - p$. Given that the first flip is a tail, the number of flips needed is equal to 1 plus a geometric random variable with parameter $p$. Thus

$$E[X] = P(Y = 1)E[X|Y = 1] + P(Y = 0)E[X|Y = 0] = p \left( 1 + \frac{1}{1-p} \right) + (1-p) \left( 1 + \frac{1}{p} \right).$$

25. (a) For $i = 1, 2, 3$, Let $Y = i$ if the first trial ends up with outcome $i$. Then


$$= p_1 E[N|Y = 1] + p_2 E[N|Y = 2] + p_3 E[N|Y = 3].$$

Given the first trial ends up with outcome $i$, $N$ is equal to 1 plus a negative binomial random variable with parameters $(2, p_i)$ (that is, sums of 2 independent geometric $p_i$ random variables). Thus

$$E[N] = p_1 (1 + \frac{2}{p_1}) + p_2 (1 + \frac{2}{p_2}) + p_3 (1 + \frac{2}{p_3}) = 7.$$

37. Let $Y = 1$ if $A$ is chosen, $Y = 2$ if $B$ is chosen and $Y = 3$ if $C$ is chosen. Then (a)

$$E[X] = \frac{1}{3}(E[X|Y = 1] + E[X|Y = 2] + E[X|Y = 3])$$

$$= \frac{1}{3} (2.6 + 3 + 3.4) = 3.$$

(b)

$$E[X^2] = \frac{1}{3} \left( E[X^2|Y = 1] + E[X^2|Y = 2] + E[X^2|Y = 3] \right)$$

$$= \frac{1}{3} (2.6 + 2.6^2 + 3 + 3^2 + 3.4 + 3.4^2)$$

$$= 3 + \frac{1}{3}(2.6^2 + 3^2 + 3.4^2)$$

and so

$$\text{Var}(X) = E[X^2] - (E[X])^2 = \frac{1}{3} (2.6^2 + 3^2 + 3.4^2) - 9.$$

38. We are given

$$f_Y(y) = \begin{cases} 
1 & y \in (0,1), \\
0 & \text{otherwise} 
\end{cases}$$
and

\[ f_{X|Y}(x|y) = \begin{cases} \frac{1}{y} & x \in (0, y), y \in (0, 1), \\ 0 & \text{otherwise} \end{cases} \]

Hence

\[ f(x, y) = \begin{cases} \frac{1}{y} & x \in (0, y), y \in (0, 1), \\ 0 & \text{otherwise} \end{cases} \]

Consequently for \( x \in (0, 1) \),

\[ f_X(x) = \int_x^1 \frac{1}{y} dy = -\ln x \]

and \( f_X(x) = 0 \) for \( x \notin (0, 1) \). So

\[ E[X] = -\int_0^1 x \ln x dx = -\frac{y^2}{4} (2 \ln y - 1) \bigg|_x^1 = \frac{1}{4} \]

\[ E[X^2] = -\int_0^1 x^2 \ln x dx = -\frac{y^3}{9} (3 \ln y - 1) \bigg|_x^1 = \frac{1}{9} \]

and

\[ \text{Var}(X) = E[X^2] - (E[X])^2 = \frac{1}{9} - \frac{1}{16}. \]

44. Let \( N \) be the number of customers entering the store on a given day. For each \( i = 1, 2, \ldots \), let \( X_i \) be the amount of money spent by customer number \( i \). The problem is implicitly assuming that \( N, X_1, X_2, \ldots \) are all independent. We are asked to find \( E[\sum_{i=1}^N X_i] \). By the argument in Example 3.10 (or going over the argument again), we have

\[ E[\sum_{i=1}^N X_i] = E[N]E[X_1] = 10 \cdot 50 = 500. \]

49. Let \( Y \) be the number of wins for \( A \) in the first 2 games.

(a) Let \( E \) be the even that \( A \) is the overall winner. Then

\[ P(E) = P(E|Y = 0)P(Y = 0) + P(E|Y = 1)P(Y = 1) + P(E|Y = 2)P(Y = 2) = 0 + P(E)2p(1-p) + p^2. \]

Thus

\[ P(E) = \frac{p^2}{1 - 2p(1-p)}. \]

(b) Let \( X \) be the number of games played. Then

\[ E[X] = E[X|Y = 0]P(Y = 0) + E[X|Y = 1]P(Y = 1) + E[X|Y = 2]P(Y = 2) = 2(1-p)^2 + (2 + E[X])2p(1-p) + 2p^2. \]
Thus

\[ E[X] = \frac{2}{1 - 2p(1 - p)}. \]

57. Let \( X \) be the number of storms in the upcoming rainy season. We are given

\[ p_{X|\Lambda}(x|\lambda) = \begin{cases} e^{-\lambda \frac{x}{x!}}, & x = 0, 1, \ldots, \lambda \in (0, 5), \\ 0, & \text{otherwise} \end{cases} \]

and

\[ f_{\Lambda}(\lambda) = \begin{cases} \frac{1}{5}, & \lambda \in (0, 5), \\ 0, & \text{otherwise} \end{cases}. \]

Thus

\[ P(X \geq 3) = \frac{1}{5} \int_{0}^{5} P(X \geq 3|\Lambda = \lambda) d\lambda = \frac{1}{5} \int_{0}^{5} (1 - (1 + \lambda + \frac{\lambda^2}{2})e^{-\lambda}) d\lambda 
= 1 - \frac{1}{5} \int_{0}^{5} (1 + \lambda + \frac{\lambda^2}{2})e^{-\lambda} d\lambda. \]

58. (a)

\[ E[N] = \int_{0}^{\infty} E[N|Y = y] f_{Y}(y) dy = \int_{0}^{\infty} y \lambda e^{-\lambda y} (\lambda y)^{r-1} \frac{\lambda e^{-\lambda y} (\lambda y)^{r-1}}{(r-1)!} dy 
= r \int_{0}^{\infty} \frac{t^r}{r!} dt = \frac{r}{\lambda}. \]

(b)

\[ E[N^2] = \int_{0}^{\infty} E[N^2|Y = y] f_{Y}(y) dy = \int_{0}^{\infty} (y + y^2) \lambda e^{-\lambda y} (\lambda y)^{r-1} \frac{\lambda e^{-\lambda y} (\lambda y)^{r-1}}{(r-1)!} dy 
= \frac{r}{\lambda} + \int_{0}^{\infty} y^2 \lambda e^{-\lambda y} (\lambda y)^{r-1} \frac{\lambda e^{-\lambda y} (\lambda y)^{r-1}}{(r-1)!} dy 
= \frac{r}{\lambda} + \frac{r(r+1)}{\lambda}. \]

Thus

\[ \text{Var}(N) = \frac{r}{\lambda} + \frac{r(r+1)}{\lambda^2} - \frac{r^2}{\lambda^2} = \frac{r}{\lambda} + \frac{r}{\lambda^2}. \]

(c)

\[ P(N = n) = \int_{0}^{\infty} P(N = n|Y = y) f_{Y}(y) dy = \int_{0}^{\infty} e^{-y} \frac{y^n \lambda e^{-\lambda y} (\lambda y)^{r-1}}{n!} \frac{\lambda e^{-\lambda y} (\lambda y)^{r-1}}{(r-1)!} dy 
= \frac{(n + r - 1)!}{n!(r-1)!} \lambda^r \int_{0}^{\infty} e^{-(\lambda + 1)y} \frac{y^n + r - 1}{(n + r - 1)!} dy 
= \frac{(n + r - 1)!}{n!(r-1)!} \lambda^r \int_{0}^{\infty} e^{-t} \frac{t^{n+r-1}}{(n + r - 1)!} dt 
= \frac{(n + r - 1)!}{n!(r-1)!} \lambda^r \frac{(\lambda + 1)^{n+r}}{(\lambda + 1)^{n+r}}. \]
63. (a) Let $E$ be the event that there is exactly one type $i$ coupon in the final collection. Let $T$ be the number of types collected before the first type $i$ coupon appears. Then $P(T = 0) = P(T = 1) = \cdots = P(T = n - 1) = \frac{1}{n}$, and for each $i = 0, 1, \ldots, n - 1$,

$$P(E|T = i) = \frac{1}{n - j}.$$ 

The last formula above is because there are still $n - i - 1$ types not collected yet when the first type $i$ is obtained and the probability starting from that point that $i$ is the last of the $n - i$ types (consisting of type $i$ and the uncollected $n - i - 1$ types) to be obtained is $1/(n - i)$. Consequently

$$P(E) = \frac{1}{n} \sum_{i=0}^{n-1} \frac{1}{n - i}.$$ 

74. For $i = 1, 2, 3, 4, 5$, let $E_i$ be the event that the $i$-component is working and let $E$ be the event that the system is working. Define a random variable $Y$ by let $Y = 1$ if the 3rd component is working and $Y = 0$ otherwise. Then

$$P(E) = p_3 P(E|Y = 1) + (1 - p_3) P(E|Y = 0).$$

$$P(E|Y = 1) = P((E_1 \cup E_2) \cap (E_4 \cup E_5)) = P(E_1 \cup E_2) P(E_4 \cup E_5)
= (p_1 + p_2 - p_1 p_2)(p_4 + p_5 - p_4 p_5)$$

and

$$P(E|Y = 0) = P((E_1 \cap E_4) \cup (E_2 \cap E_5)) = P(E_1 \cap E_4) + P(E_2 \cap E_5) - P(E_1 \cap E_4 \cap E_2 \cap E_5)
= p_1 p_4 + p_2 p_5 - p_1 p_2 p_4 p_5.$$ 

Thus

$$P(E) = p_3 (p_1 + p_2 - p_1 p_2)(p_4 + p_5 - p_4 p_5)
+ (1 - p_3) (p_1 p_4 + p_2 p_5 - p_1 p_2 p_4 p_5).$$

92. (a) Let $N$ be the number of coins that Josh spots. We are given that $N$ is a Poisson random variable with parameter 6. Let $X$ be the amount of money in cents that Josh picks up, let $A$ be the number of nickels that Josh picks up, let $A$ be the number of dimes that Josh picks up and let $A$ be the number of quarters that Josh picks up. Then $X = 5A + 10B + 25C$. Thus


$$E[A] = \sum_{n=0}^{\infty} E[A|N = n] P(N = n) = \sum_{n=0}^{\infty} \frac{n}{4} P(N = n)
= \frac{1}{4} \sum_{n=0}^{\infty} n P(N = n) = \frac{1}{4} E[N] = \frac{3}{2}.$$
Similarly, we also have $E[B] = E[C] = \frac{3}{2}$. Thus

$$E[X] = 40 \cdot \frac{3}{2} = 60.$$  

94. (a) Regard $b$ as fixed and let $HG(w, r)$ with parameters $w$ and $r$.

$$P(M - 1 = n) = P(M = n + 1) = \frac{(n + 1)P(N = n + 1)}{E[N]}$$

$$= \frac{(n + 1)\binom{w}{n + 1} \binom{b}{r - n - 1} w + b}{\binom{w + b}{r}}$$

$$= \frac{\binom{w - 1}{n} \binom{b}{(r - 1) - n}}{\binom{(w - 1) + b}{r - 1}}.$$  

Thus $M - 1$ is an $HG(w - 1, r - 1)$ random variable.

(b) Using (a) and Corollary 3.6, we get the following recursive equation:

$$P_{w,r}(k) = \frac{1}{k w + b} \sum_{j=1}^{k} j \alpha_j P_{w-1,r-1,k-j}.$$  

(c)

$$P_{w,r}(0) = P(N = 0) = \binom{b}{w + b} \binom{r}{r}$$

$$P_{w,r}(1) = \frac{rw}{w + b} \alpha_1 P_{w-1,r-1}(0) = \frac{rw}{w + b} \binom{b}{w - 1 + b} \binom{r}{r - 1} \alpha_1$$

$$P_{w,r}(2) = \frac{1}{2} \frac{rw}{w + b} \left( \alpha_1 P_{w-1,r-1}(1) + 2 \alpha_2 P_{w-1,r-1}(0) \right)$$

$$= \frac{1}{2} \frac{rw}{w + b} \frac{(r - 1)(w - 1)}{w - 1 + b} \binom{b}{w - 2 + b} \binom{r}{r - 1} \alpha_1$$

$$+ \frac{rw}{w + b} \binom{b}{w - 1 + b} \binom{r - 1}{w - 1 + b} \alpha_2.$$  

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