# Sketch of solutions to HW2

Chapter 2

42. Let $X_1 = 1$. Let $X_2$ be the number number of additional coupons needed, after the first coupon, to get a new type, and let $X_3$ be the number of additional coupons needed, after two types has been collected, to get a new type, etc. Then $X_i$ is a geometric random variable with parameter $1 - \frac{1}{m} = \frac{m-1}{m}$, $X_3$ is a geometric random variable with parameter $\frac{m-2}{m}$, etc. We also have that $X_1 + X_2 + \cdots + X_m$ is the number of coupons needed in order to get a complete set. Thus

$$\begin{aligned} E[X_1 + X_2 + \cdots + X_m] &= 1 + \frac{m}{m-1} + \frac{m}{m-2} + \cdots + m \\ &= m(1 + \frac{1}{2} + \cdots \frac{1}{m}). \end{aligned}$$

51. Let $X_1$ be the number of flips needed for the first head, let $X_2$ be the number of additional flips for the second head, etc. Then $N = X_1 + \cdots X_r$ is the number of flips needed to get $r$ heads. Each $X_i$ is a geometric random variable with parameter $p$.

$$E[N] = \frac{r}{p}.$$

53.

$$E[X^n] = \int_0^1 x^n dx = \frac{1}{n+1}.$$

$$E[X^{2n}] = \int_0^1 x^{2n} dx = \frac{1}{2n+1}.$$

$$\text{Var}(X^n) = E[X^{2n}] - (E[X^n])^2 = \frac{1}{2n+1} - \frac{1}{(n+1)^2}.$$

56. For $i = 1, \cdots, n$, let $X_i = 1$ if there is at least one type $i$ coupon in the collection of $k$ coupons and $X_i = 0$ otherwise. Then $X_1 + \cdots + X_n$ is the number of distinct types in the collection of $k$ coupons. For $i = 1, \cdots, n$,

$$P(X_i = 0) = \left(1 - \frac{1}{n}\right)^k, \quad P(X_i = 1) = 1 - \left(1 - \frac{1}{n}\right)^k.$$

For $i \neq j$,

$$P(X_i X_j = 0) = P(X_i = 0) + P(X_j = 0) - P(X_i = 0, X_j = 0) = 2\left(1 - \frac{1}{n}\right)^k - \left(1 - \frac{2}{n}\right)^k,$$

$$P(X_i X_j = 1) = 1 - 2\left(1 - \frac{1}{n}\right)^k + \left(1 - \frac{2}{n}\right)^k.$$

Thus

$$E[X_i] = 1 - \left(1 - \frac{1}{n}\right)^k, \quad \mathrm{Var}(X_i) = \left(1 - \frac{1}{n}\right)^k \left(1 - \left(1 - \frac{1}{n}\right)^k\right),$$

and

$$\mathrm{Cov}(X_i, X_j) = 1 - 2\left(1 - \frac{1}{n}\right)^k + \left(1 - \frac{2}{n}\right)^k - \left(1 - \left(1 - \frac{1}{n}\right)^k\right)^2.$$

Consequently

$$E[X_1 + \cdots + X_n] = n\left(1 - \left(1 - \frac{1}{n}\right)^k\right)$$

and

$$\mathrm{Var}(X_1 + \cdots + X_n) = n\left(1 - \frac{1}{n}\right)^k \left(1 - \left(1 - \frac{1}{n}\right)^k\right)$$
$$+ n(n-1)\left(1 - 2\left(1 - \frac{1}{n}\right)^k + \left(1 - \frac{2}{n}\right)^k - \left(1 - \left(1 - \frac{1}{n}\right)^k\right)^2\right).$$

60. $E[XY] = \mu_x\mu_y$, $E[X^2Y^2] = E[X^2]E[Y^2] = (\sigma_x^2 + \mu_x^2)(\sigma_y^2 + \mu_y^2)$. From these the desired result follows.

65. (a) The probability that there are two accidents in a given day is $e^{-2}\frac{2^2}{2!} = 2e^{-2}$. Thus the probability that 3 of the next 5 days have two accidents is

$$\binom{5}{3}\left(2e^{-2}\right)^3\left(1 - 2e^{-2}\right)^2.$$

(b) The number of accidents in the next two days is a Poisson random variable with parameter 4. Thus there are a total of 6 accidents in the next 2 days is $e^{-4}\frac{4^6}{6!}$.

(c) The number of major accidents in a given day is a Poisson random variable with parameter $2p$. Thus the probability that there are no major accidents tomorrow is $e^{-2p}$.

72. See the end of the book.

## Chapter 3

3.

$$E[X|Y=1] = 1 \cdot p_{X|Y}(1|1) + 2 \cdot p_{X|Y}(2|1) + 3 \cdot p_{X|Y}(3|1) = \frac{1}{5} + 2 \cdot \frac{3}{5} + 3 \cdot \frac{1}{5}.$$

The other two are similar.

5.

$$p_{X|Y}(0|3) = \frac{\binom{6}{3}\binom{5}{3}}{\binom{14}{6}}, \quad p_{X|Y}(1|3) = \frac{3\binom{6}{2}\binom{5}{3}}{\binom{14}{6}}$$

and

$$p_{X|Y}(2|3) = \frac{\binom{3}{2} \cdot 6 \cdot \binom{5}{3}}{\binom{14}{6}}, \quad p_{X|Y}(3|3) = \frac{\binom{3}{3}\binom{5}{3}}{\binom{14}{6}}$$

One can similarly write down the conditional mass function of $X$ given $Y = 1$. From that one can easily find $E[X|Y = 1]$.

11. For any $y \in (0, \infty)$,

$$f_Y(y) = \int_{-y}^{y} \frac{y^2 - x^2}{8} e^{-y} dx = \frac{5}{24} y^3 e^{-y}.$$

Thus for $x \in (-y, y)$,

$$f_{X|Y}(x|y) = \frac{3(y^2 - x^2)}{5y^3},$$

and for $x \notin (-y, y)$, $f_{X|Y}(x|y) = 0$. Thus

$$E[X|Y = y] = \int_{-y}^{y} x \frac{3(y^2 - x^2)}{5y^3} dx = \frac{3}{5y} \int_{-y}^{y} x dx - \frac{3}{5y^3} \int_{-y}^{y} x^3 dx = 0.$$

12. For $y \in (0, \infty)$,

$$F_Y(y) = \int_0^{\infty} \frac{1}{y} e^{-x/y} e^{-y} dx = e{-}y.$$

Thus

$$F_{X|Y}(x|y) = \frac{1}{y} e^{-x/y}, \qquad x > 0$$

which is an exponential density with parameter $\frac{1}{y}$. Thus $E[X|Y = y] = y$.

15. For $y \in (0, \infty)$,

$$f_Y(y) = \int_0^{y} \frac{1}{y} e^{-y} dx = e^{-y}$$

and

$$f_{X|Y}(x|y) = \frac{1}{y}, \qquad x \in (0, y).$$

Thus

$$E[X^2|Y = y] = \int_0^{y} x^2 \frac{1}{y} dx = \frac{y^2}{3}.$$