Sketch of solutions to HW11

Chapter 9

15. (b) The minimal path sets are $\{1,2,4,6\}$ and $\{1,3,5,6\}$. The minimal cut sets are $\{1\}$, $\{6\}$, $\{2,3\}$, $\{2,5\}$, $\{4,3\}$, $\{4,5\}$. Thus by (9.13), the upper and lower bounds are

$$\frac{1}{2}\cdot\frac{1}{2}\cdot\left(1-\frac{1}{4}\right)^4 \le r(\mathbf{p}) \le 1-\left(1-\frac{1}{16}\right)^2$$

The exact value is

$$r(\mathbf{p}) = \frac{1}{2}\cdot\left(\frac{1}{4}+\frac{1}{4}-\frac{1}{16}\right)\cdot\frac{1}{2}.$$

(c) The minimal path sets are $\{1,4\}$ and $\{2,3,4\}$. The minimal cut sets are $\{1,2\}$, $\{1,3\}$ and $\{4\}$. Thus by (9.13), the upper and lower bounds are

$$\left(1-\frac{1}{4}\right)\cdot\left(1-\frac{1}{4}\right)\cdot\left(1-\frac{1}{2}\right) \le r(\mathbf{p}) \le 1-\left(1-\frac{1}{4}\right)\left(1-\frac{1}{8}\right)$$

The exact value is

$$r(\mathbf{p}) = \left(\frac{1}{2}+\frac{1}{4}-\frac{1}{8}\right)\cdot\frac{1}{2}.$$

16. (a) The minimal path sets are $\{1,2\}$, $\{1,3\}$ and $\{2,3\}$. Using these, we get the following upper and lower bounds for $r(\mathbf{p})$:

$$3p^2-3p^3 \le r(\mathbf{p}) \le 3p^2. \tag{1}$$

The minimal cut sets are $\{1,2\}$, $\{1,3\}$ and $\{2,3\}$. Using these, we get the following upper and lower bounds for $r(\mathbf{p})$:

$$1-3(1-p)^2 \le r(\mathbf{p}) \le 1-3(1-p)^2+3(1-p)^3. \tag{2}$$

Using method 2, we get the following upper and lower bounds for $r(\mathbf{p})$:

$$(1-(1-p)^2)^3 \le r(\mathbf{p}) \le 1-(1-p^2)^3. \tag{3}$$

(b) The minimal path sets are $\{1,2\}$, $\{1,3\}$, $\{1,4\}$, $\{2,3\}$, $\{2,4\}$ and $\{3,4\}$. Using these, we get the following upper and lower bounds for $r(\mathbf{p})$:

$$6p^2-12p^3-3p^4 \le r(\mathbf{p}) \le 6p^2. \tag{4}$$

The minimal cut sets are $\{1,2,3\}$, $\{1,2,4\}$, $\{1,3,4\}$ and $\{2,3,4\}$. Using these, we get the following upper and lower bounds for $r(\mathbf{p})$:

$$1-4(1-p)^3 \le r(\mathbf{p}) \le 1-4(1-p)^3+6(1-p)^4. \tag{5}$$

Using method 2, we get the following upper and lower bounds for $r(\mathbf{p})$:

$$(1-(1-p)^3)^4 \le r(\mathbf{p}) \le 1-(1-p^2)^6. \tag{6}$$

(c) Just plug in the values and compare.

18. (a) The minimal cut sets are $\{1,4\}$, $\{1,5\}$, $\{2,4\}$, $\{2,5\}$ and $\{3\}$.
(b) The reliability function of the system is

$$r(\mathbf{p}) = p_1p_2p_3 + p_3p_4p_5 - p_1p_2p_3p_4p_5.$$

For $i = 1, \cdots, 5$,

$$\overline{F_i}(t) = \begin{cases} 1-t & t \in [0,1], \\ 0 & t > 1. \end{cases}$$

Thus the distribution F of the system lifetime is determined by

$$\overline{F}(t) = \begin{cases} 2(1-t)^3 - (1-t)^5 & \in [0,1], \\ 0 & t > 1. \end{cases}$$

Thus the probability that the system lifetime will be less than $\frac{1}{2}$ is

$$1 - \overline{F}(\frac{1}{2}) = 2(1-\frac{1}{2})^3 - (1-\frac{1}{2})^5 = \frac{1}{4} - \frac{1}{32}.$$

21 (a). (i), (ii), (iv). The reason for (iv) is that the system is a two-of-three system.

28. We have

$$\overline{F_1}(t) = \begin{cases} 1-t & t \in [0,1], \\ 0 & t > 1. \end{cases}$$

and

$$\overline{F_2}(t) = \begin{cases} 1-\frac{t}{2} & t \in [0,2], \\ 0 & t > 2. \end{cases}$$

For a series system, the reliability function is $r(\mathbf{p}) = p_1p_2$. Thus in this case we have

$$\overline{F}(t) = \begin{cases} \frac{1}{2}(1-t)(2-t) & t \in [0,1], \\ 0 & t > 1, \end{cases}$$

and the expected system lifetime is

$$\frac{1}{2}\int_0^1 (1-t)(2-t)dt.$$

For a parallel system, the reliability function is $r(\mathbf{p}) = 1-(1-p_1)(1-p_2)$. Thus in this case we have

$$\overline{F}(t) = \begin{cases} 1-\frac{t^2}{2} & t \in [0,1], \\ 1-\frac{t}{2} & t \in (1,2], \\ 0 & t > 2, \end{cases}$$

and the expected system lifetime is

$$\int_0^1 \left(1 - \frac{t^2}{2}\right) dt + \int_1^2 \left(1 - \frac{t}{2}\right) dt.$$

29. The reliability function is $r(\mathbf{p}) = 1-(1-p_1)(1-p_2)$. By assumption we have $\overline{F_1}(t) = e^{-\mu_1 t}$ and $\overline{F_2}(t) = e^{-\mu_2 t}$. Thus

$$\overline{F}(t) = 1 - (1 - e^{-\mu_1 t})1 - (e^{-\mu_2 t}) = e^{-\mu_1 t} + e^{-\mu_2 t} - e^{-(\mu_1+\mu_2)t}.$$

The mean system lifetime is

$$\int_0^\infty (e^{-\mu_1 t} + e^{-\mu_2 t} - e^{-(\mu_1+\mu_2)t})dt = \frac{1}{\mu_1} + \frac{1}{\mu_2} - \frac{1}{\mu_1+\mu_2} = \frac{1}{\mu_1+\mu_2} + \frac{\mu_1}{(\mu_1+\mu_2)\mu_2} + \frac{\mu_2}{(\mu_1+\mu_2)\mu_1}.$$

30. See the solution at the end of book.