Sketch of solutions to HW1

Chapter 1

25. (a) The answer is
\[
\frac{52 \cdot 3}{52 \cdot 51} = \frac{3}{51}.
\]
(b) The probability that they are of different suits is
\[
\frac{52 \cdot 39}{52 \cdot 51} = \frac{39}{51},
\]
so the answer is \(3/39 = 1/13\).

30. Let \(B\) be the event that Bill hits his target, and \(G\) be the event that George hits his target.

(a) The probability that exactly one shot hits the target is
\[
P(B \cap G^c) + P(G \cap B^c) = P(B)P(G^c) + P(G)P(B^c)
= (.7)(.6) + (.4)(.3).
\]
Thus the answer is
\[
\frac{(.4)(.3)}{(.7)(.6) + (.4)(.3)} = \frac{2}{9}.
\]

32. For \(i = 1, \cdots, n\), let \(E_i\) be the event that the \(i\)th man gets his hat. Then
\[
P(E_i) = \frac{(n-1)!}{n!} = \frac{1}{n}, \quad i = 1, \cdots, n.
\]
For any \(1 \leq i_1 < i_2 \leq n\),
\[
P(E_{i_1} \cap E_{i_2}) = \frac{(n-2)!}{n!} = \frac{1}{n(n-1)}.
\]
For \(k = 3, \cdots, n\), and \(1 < i_1 < i_2 < \cdots < i_k \leq n\),
\[
P(\cap_{i=1}^k E_{i_j}) = \frac{(n-k)!}{n!}.
\]
Thus by the inclusion-exclusion formula
\[
P(\cup_{i=1}^n E_i) = \sum_{i=1}^n P(E_i) - \sum_{i_1 < i_2} P(E_{i_1} \cap E_{i_2}) + \cdots
+ (-1)^{k+1} \sum_{i_1 < i_2 < \cdots < i_k} P(\cap_{j=1}^k E_{i_j}) + \cdots + (-1)^{n+1} P(\cap_{i=1}^n E_i)
= n \cdot \frac{1}{n} - \binom{n}{2} \cdot \frac{(n-2)!}{n!} + \cdots
+ (-1)^{k+1} \binom{n}{k} \frac{(n-k)!}{n!} + \cdots + (-1)^{n+1} \frac{1}{n!}
= 1 - \frac{1}{2!} + \frac{1}{3!} - \frac{1}{4!} + \cdots + (-1)^{n+1} \frac{1}{n!}.
\]
Therefore the probability that none of the n men selects his own hat is
\[ 1 - P(\cup_{i=1}^{n} E_i) = \frac{1}{2!} - \frac{1}{3!} + \cdots + (-1)^n \frac{1}{n!}. \]

35. (a) \( P(\text{HHHH}) = p^4. \)
(b) \( P(\text{PTHHH}) = p^3(1 - p). \)
(c) The pattern HHHH can only occur before THHH if the first four coin flips come up heads. Hence, \( P(\text{THHH}) \) occurs before HHHH \( = 1 - p^4. \)

36. Let \( E_1 \) be the event that Box 1 is selected, \( E_2 \) the even that Box 2 is selected. Let \( B \) be the event that the ball is black. Then
\[ P(B) = P(E_1 \cap B) + P(E_2 \cap B) = P(E_1)P(B|E_1) + P(E_2)P(B|E_2) \]
\[ = \frac{1}{2} \cdot \frac{1}{2} + \frac{1}{2} \cdot \frac{2}{2} = \frac{7}{12}. \]

38. Let \( W_1 \) be the event that the transferred ball is white, \( B_1 \) the event that the transferred ball is black, \( W_2 \) the event that the ball drawn from Urn II is white. Then
\[ P(W_2) = P(W_1 \cap W_2) + P(B_1 \cap W_2) = P(W_1)P(W_2|W_1) + P(B_1)P(W_2|B_1) \]
\[ = \frac{2}{3} \cdot \frac{2}{7} + \frac{1}{3} \cdot \frac{1}{7} = \frac{5}{21}. \]
Thus
\[ P(W_1|W_2) = \frac{P(W_1 \cap W_2)}{P(W_2)} = \frac{4/21}{5/21} = \frac{4}{5}. \]

42. Let \( E_1 \) be the event that the two-headed coin is selected, \( E_2 \) the event that the fair coin is selected, and \( E_3 \) the biased coin that comes up heads 75 percent of time is selected. Let \( H \) be the event that the coin comes up heads. Then
\[ P(H) = P(E_1 \cap H) + P(E_2 \cap H) + P(E_3 \cap H) \]
\[ = P(E_1)P(H|E_1) + P(E_2)P(H|E_2) + P(E_3)P(H|E_3) \]
\[ = \frac{1}{3} \cdot \frac{1}{3} + \frac{1}{2} \cdot \frac{1}{3} + \frac{3}{4} \cdot \frac{3}{4} \]
\[ = \frac{9}{12}. \]
Thus
\[ P(E_1|H) = \frac{P(E_1 \cap H)}{P(H)} = \frac{1/3}{9/12} = \frac{4}{9}. \]

Chapter 2
8. The probability mass function of $X$ is

$$p(x) = \begin{cases} \frac{1}{2}, & x = 0 \\ \frac{1}{2}, & x = 1 \\ 0, & \text{otherwise} \end{cases}$$

9. The probability mass function of $X$ is

$$p(x) = \begin{cases} \frac{1}{2}, & x = 0 \\ \frac{1}{10}, & x = 1 \\ \frac{1}{5}, & x = 2 \\ \frac{1}{10}, & x = 3 \\ \frac{1}{10}, & x = 3.5 \\ 0, & \text{otherwise} \end{cases}$$

16. The desired probability is equal to

$$1 - P(\text{exactly 51 show up}) - P(\text{all 52 show up}) = 1 - 52 \cdot (.95)^{51}(.05) - (.95)^{52}$$

23. For $n \geq r$, $X = n$ is equivalent to that the $n$-th flip is heads and there are exactly $r - 1$ heads in the first $n - 1$ flips. Thus

$$P(X = n) = \binom{n-1}{r-1} p^{r-1} (1-p)^{n-r} \cdot p$$

$$= \binom{n-1}{r-1} p^r (1-p)^{n-r}.$$ 

34. (a) Since

$$\int_0^2 (4x - 2x^2)dx = 8 - \frac{16}{3} = \frac{8}{3},$$

we have $c = \frac{3}{8}$.

(b) $P(\frac{1}{2} < X < \frac{3}{2}) = \frac{3}{8} \int_{1/2}^{3/2} (4x - 2x^2)dx$.

39.

$$E[X] = 1 \cdot \frac{1}{2} + 2 \cdot \frac{1}{3} + 24 \cdot \frac{1}{6}.$$
40. Let \( X \) be the number of games that are played. Then

\[
P(X = 4) = p^4 + (1 - p)^4
\]
\[
P(X = 5) = \binom{4}{3} p^4(1 - p) + \binom{4}{3} p(1 - p)^4
\]
\[
P(X = 6) = \binom{5}{3} p^4(1 - p)^2 + \binom{5}{3} p^2(1 - p)^4
\]
\[
P(X = 7) = \binom{6}{3} p^4(1 - p)^3 + \binom{6}{3} p^3(1 - p)^4,
\]

Thus

\[
E[X] = 4 \left( p^4 + (1 - p)^4 \right) + 5 \left( \binom{4}{3} p^4(1 - p) + \binom{4}{3} p(1 - p)^4 \right)
6 \left( \binom{5}{3} p^4(1 - p)^2 + \binom{5}{3} p^2(1 - p)^4 \right) + 7 \left( \binom{6}{3} p^4(1 - p)^3 + \binom{6}{3} p^3(1 - p)^4 \right).
\]

When \( p = \frac{1}{2} \),

\[
E[X] = 4 \cdot 2 \cdot 2^{-4} + 5 \cdot 2 \left( \binom{4}{3} 2^{-5} + 6 \cdot 2 \left( \binom{5}{3} 2^{-6} + 7 \cdot 2 \left( \binom{6}{3} 2^{-7}\right)\right)\right).
\]