Sketch of solutions to HW1

Chapter 1

25. (a) The answer is

$$\frac{52\cdot 3}{52\cdot 51}=\frac{3}{51}.$$

(b) The probability that they are of different suits is

$$\frac{52\cdot 39}{52\cdot 51}=\frac{39}{51},$$

so the answer is $3/39=1/13$.

30. Let B be the event that Bill hits his target, and G be the event that George hits his target.

(a) The probability that exactly one shot hits the target is

$$\begin{aligned}P(B\cap G^c)+P(G\cap B^c)&=P(B)P(G^c)+P(G)P(B^c)\\&=(.7)(.6)+(.4)(.3).\end{aligned}$$

Thus the answer is

$$\frac{(.4)(.3)}{(.7)(.6)+(.4)(.3)}=\frac{2}{9}.$$

32. For $i=1,\cdots,n$, let E_i be the event that the ith man gets his hat. Then

$$P(E_i)=\frac{(n-1)!}{n!}=\frac{1}{n},\qquad i=1,\cdots,n.$$

For any $1\le i_1<i_2\le n$,

$$P(E_{i_1}\cap E_{i_2})=\frac{(n-2)!}{n!}=\frac{1}{n(n-1)}.$$

For $k=3,\cdots,n$, and $1<i_1<i_2<\cdots<i_k\le n$,

$$P(\cap_{j=1}^k E_{i_j})=\frac{(n-k)!}{n!}.$$

Thus by the inclusion-exclusion formula

$$\begin{aligned}P(\cup_{i=1}^n E_i)=&\sum_{i=1}^n P(E_i)-\sum_{i_1<i_2}P(E_{i_1}\cap E_{i_2})+\cdots\\&+(-1)^{k+1}\sum_{i_1<i_2<\cdots<i_k}P(\cap_{j=1}^k E_{i_j})+\cdots+(-1)^{n+1}P(\cap_{i=1}^n E_i)\\=&\,n\cdot\frac{1}{n}-\binom{n}{2}\cdot\frac{(n-2)!}{n!}+\cdots\\&+(-1)^{k+1}\binom{n}{k}\frac{(n-k)!}{n!}+\cdots+(-1)^{n+1}\frac{1}{n!}\\=&\,1-\frac{1}{2!}+\frac{1}{3!}-\frac{1}{4!}+\cdots+(-1)^{n+1}\frac{1}{n!}.\end{aligned}$$

Therefore the probability that none of the n men selects his own hat is

$$1 - P(\cup_{i=1}^{n} E_i) = \frac{1}{2!} - \frac{1}{3!} + \cdots + (-1)^n \frac{1}{n!}.$$

35. (a) $P(HHHH) = p^4$.
(b) $PTHHH) = p^3(1-p)$.
(c) The pattern $HHHH$ can *only* occur before $THHH$ if the first four coin flips come up heads. Hence, $PTHHH$ occurs before $HHHH) = 1 - p^4$.

36. Let E_1 be the event that Box 1 is selected, E_2 the even that Box 2 is selected. Let B be the event that the ball is black. Then

$$\begin{aligned} P(B) &= P(E_1 \cap B) + P(E_2 \cap B) = P(E_1)P(B|E_1) + P(E_2)P(B|E_2) \\ &= \frac{1}{2} \cdot \frac{1}{2} + \frac{1}{2} \cdot \frac{2}{3} = \frac{7}{12} \end{aligned}$$

38. Let W_1 be the event that the transferred ball is white, B_1 the event that the transferred ball is black, W_2 the event that the ball drawn from Urn II is white. Then

$$\begin{aligned} P(W_2) &= P(W_1 \cap W_2) + P(B_1 \cap W_2) = P(W_1)P(W_2|W_1) + P(B_1)P(W_2|B_1) \\ &= \frac{2}{3} \cdot \frac{2}{7} + \frac{1}{3} \cdot \frac{1}{7} = \frac{5}{21}. \end{aligned}$$

Thus

$$P(W_1|W_2) = \frac{P(W_1 \cap W_2)}{P(W_2)} = \frac{4/21}{5/21} = \frac{4}{5}.$$

42. Let E_1 be the event that the two-headed coin is selected, E_2 the event that the fair coin is selected, and E_3 the biased coin that comes up heads 75 percent of time is selected. Let H be the event that the coin comes up heads. Then

$$\begin{aligned} P(H) &= P(E_1 \cap H) + P(E_2 \cap H) + P(E_3 \cap H) \\ &= P(E_1)P(H|E_1) + P(E_2)P(H|E_2) + P(E_3)P(H|E_3) \\ &= \frac{1}{3} \cdot 1 + \frac{1}{3} \cdot \frac{1}{2} + \frac{1}{3} \cdot \frac{3}{4} \\ &= \frac{9}{12}. \end{aligned}$$

Thus

$$P(E_1|H) = \frac{P(E_1 \cap H)}{P(H)} = \frac{1/3}{9/12} = \frac{4}{9}.$$

Chapter 2

8. The probability mass function of X is

$$p(x) = \begin{cases} \frac{1}{2}, & x = 0 \\ \frac{1}{2}, & x = 1 \\ 0, & \text{otherwise.} \end{cases}$$

9. The probability mass function of X is

$$p(x) = \begin{cases} \frac{1}{2}, & x = 0 \\ \frac{1}{10}, & x = 1 \\ \frac{1}{5}, & x = 2 \\ \frac{1}{10}, & x = 3 \\ \frac{1}{10}, & x = 3.5 \\ 0, & \text{otherwise.} \end{cases}$$

16. The desired probability is equal to

$$\begin{aligned} &1 - P(\text{ eaxctly 51 show up}) - P(\text{ all 52 show up }) \\ &= 1 - 52 \cdot (.95)^{51}(.05) - (.95)^{52} \end{aligned}$$

23. For $n \geq r$, $X = n$ is equivalent to that the n-th flip is heads and there are exactly $r - 1$ heads in the first $n - 1$ flips. Thus

$$\begin{aligned} P(X = n) &= \binom{n-1}{r-1} p^{r-1}(1-p)^{n-r} \cdot p \\ &= \binom{n-1}{r-1} p^{r}(1-p)^{n-r}. \end{aligned}$$

34. (a) Since

$$\int_0^2 (4x - 2x^2)dx = 8 - \frac{16}{3} = \frac{8}{3},$$

we have $c = \frac{3}{8}$.

(b)

$$P(\frac{1}{2} < X < \frac{3}{2}) = \frac{3}{8} \int_{1/2}^{3/2} (4x - 2x^2)dx.$$

39.

$$E[X] = 1 \cdot \frac{1}{2} + 2 \cdot \frac{1}{3} + 24 \cdot \frac{1}{6}.$$

40. Let X be the number of games that are played. Then

$$\begin{aligned}
P(X=4) &= p^4 + (1-p)^4 \\
P(X=5) &= \binom{4}{3} p^4(1-p) + \binom{4}{3} p(1-p)^4 \\
P(X=6) &= \binom{5}{3} p^4(1-p)^2 + \binom{5}{3} p^2(1-p)^4 \\
P(X=7) &= \binom{6}{3} p^4(1-p)^3 + \binom{6}{3} p^3(1-p)^4,
\end{aligned}$$

Thus

$$\begin{aligned}
E[X] = 4\left(p^4 + (1-p)^4\right) + 5\left(\binom{4}{3} p^4(1-p) + \binom{4}{3} p(1-p)^4\right) \\
6\left(\binom{5}{3} p^4(1-p)^2 + \binom{5}{3} p^2(1-p)^4\right) + 7\left(\binom{6}{3} p^4(1-p)^3 + \binom{6}{3} p^3(1-p)^4\right).
\end{aligned}$$

When $p = \frac{1}{2}$,

$$E[X] = 4 \cdot 2 \cdot 2^{-4} + 5 \cdot 2\binom{4}{3} 2^{-5} + 6 \cdot 2\binom{5}{3} 2^{-6} + 7 \cdot 2\binom{6}{3} 2^{-7}.$$