Solutions to Math 490 Test 3, Spring 2018

Calculators, books, notes and extra papers are *not* allowed on this test!

Show all work to qualify for full credits

1. (20 points) Let $\{N(t) : t \geq 0\}$ be a Poisson process with rate λ, and T be a non-negative random variable with mean μ and variance σ^2. Suppose that $\{N(t) : t \geq 0\}$ and T are independent. Find $\text{Cov}(N(T), T)$.

Solution: Note that for $n = 0, 1, \ldots,$

$$E[N(T)|T = n] = E[N(n)|T = n] = E[N(n)] = n\lambda,$$

we have $E[N(T)|T] = T\lambda$. Thus $E[N(T)] = E[E[N(T)|T]] = \mu\lambda$. We have $E[TN(T)|T] = TE[N(T)|T] = T^2\lambda$. Thus

$$E[TN(T)] = E[E[TN(T)|T]] = E[T^2]\lambda = (\sigma^2 + \mu^2)\lambda.$$

Therefore

$$\text{Cov}(N(T), T) = E[TN(T)] - E[T]E[N(T)] = (\sigma^2 + \mu^2)\lambda - \mu^2\lambda = \sigma^2\lambda.$$

2. (20 points) An insurance company pays out claims in accordance with a Poisson process having rate $\lambda = 5$ per week. Suppose that claim amounts are independent random variables. If the amount of money paid in each claim is an exponential random variable with mean \$200, find the mean and variance of the amount of money paid by the company in a four-week period.

Solution: The mean is

$$5 \cdot 4 \cdot 200 = 4000.$$

The variance is

$$5 \cdot 4 \cdot 80000 = 1600000.$$

3. (20 points) A barber shop with a single barber has room for at most two customers. Potential customers arrive at a Poisson rate of 3 per hour, the successive service times are independent random variables with mean $\frac{1}{3}$ hour. In the long run, (a) what is the average number of customers in the shop? (b) what is the proportion of potential customers that enter the shop?

Solution: With the number of customers in the shop as the state, we get a birth and death process with

$$\lambda_0 = \lambda_1 = 3, \mu_1 = \mu_2 = 3$$

Therefore

$$P_1 = P_0, \quad P_2 = P_1 = P_0.$$

Hence

$$P_0 = \frac{1}{3}.$$

(a) The average number of customers in the shop is

$$P_1 + 2P_2 = 1.$$

(b) The proportion of customers that enter the shop is

$$\frac{\lambda(1-P_2)}{\lambda} = 1 - P_2 = \frac{2}{3}.$$

4. (20 points) For the structure given below, (a) give the minimal path sets; (b) give the minimal cut sets; (c) give the reliability function.

Solution: (a) The minimal paths sets are $\{1,3,5\}$, $\{1,3,6\}$, $\{2,4,5\}$, $\{2,4,6\}$.
(b) The minimal cut sets are $\{1,2\}$, $\{1,4\}$, $\{3,2\}$, $\{3,4\}$, $\{5,6\}$.
(c) The reliability function is

$$\begin{aligned} r(p) &= P(\text{either } X_1X_3 = 1 \text{ or } X_2X_4 = 1)P(\text{either } X_5 = 1 \text{ or } X_6 = 1) \\ &= (p_1p_3 + p_2p_4 - p_1p_3p_2p_4)(p_5 + p_6 - p_5p_6) \end{aligned}$$