Solutions to Math 490 Test 2, Spring 2018

Calculators, books, notes and extra papers are *not* allowed on this test!

Show all work to qualify for full credits

1. (20 points) In a good weather year the number of storms is Poisson distributed with mean 2; in a bad year it is Poisson distributed with mean 4. Suppose that any year's weather condition depends on past years only through the previous year's condition. Suppose that a good year is twice as likely to be followed by a bad year as by a good year, and that a bad year is equally likely to be followed by a bad year or a good year. Find the long-run average number of storms per year.

Solution: Letting 0 stand for a good year and 1 for a bad year, the successive states follow a Markov chain with transition probability matrix

$$P = \begin{pmatrix} \frac{1}{3} & \frac{2}{3} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}.$$

By solving the system

$$\begin{aligned} \pi_0 &= \frac{1}{3}\pi_0 + \frac{1}{2}\pi_1 \\ \pi_1 &= \frac{2}{3}\pi_0 + \frac{1}{2}\pi_1 \\ \pi_0 + \pi_1 &= 1, \end{aligned}$$

we get $\pi_0 = \frac{3}{7}$, $\pi_1 = \frac{4}{7}$. Thus the long-run average number of storms is $2\frac{3}{7} + 4\frac{4}{7} = \frac{22}{7}$.

2. (20 points) Consider a branching process X_n with offspring distribution given by $P_0 = 1/6$, $P_1 = 1/2$ and $P_3 = 1/3$. Find the extinction probability π_0 given that $X_0 = 1$.

Solution: Solving the equation

$$\pi = \frac{1}{6} + \pi\frac{1}{2} + \pi^3\frac{1}{3},$$

we get only two positive solutions which are $(\sqrt{3} - 1)/2$ and 1. So $\pi_0 = (\sqrt{3} - 1)/2$.

3. (20 points) Let X_1 and X_2 be independent exponential random variables with parameter $\lambda = 2$. Let $Y = \max(X_1, X_2)$. Find the mean and variance of Y.

Solution: Let $X = \min(X_1, X_2)$. Then X is an exponential random variable with parameter 4. By the memoryless property of exponential random variables, $Y - X$ is an exponential random variable with parameter 2 and is independent of X. Thus

$$E[Y] = E[X] + E[Y - X] = \frac{1}{4} + \frac{1}{2} = \frac{3}{4}$$

and

$$\mathrm{Var}(Y) = \mathrm{Var}(X) + \mathrm{Var}(Y - X) = \frac{1}{16} + \frac{1}{4} = \frac{5}{16}.$$

4. (20 points) Let $\{N(t), t \geq 0\}$ be a Poisson process with rate $\lambda = 3$. Let S_n be the time of the n-th incident. Find (a) $E[S_5|N(1) = 2]$; and (b) $E[N(5) - N(2)|N(1) = 2]$.

Solution: (b) Given $N(1) = 2$, the conditional distribution of S_5 is the same as the unconditional distribution of $1 + S_3$. Thus

$$E[S_5|N(1) = 2] = 1 + \frac{3}{\lambda} = 2.$$

(c) By using the independent increment property and the stationary increments property,

$$E[N(5) - N(2)|N(1) = 2] = E[N(5) - N(2)] = E[N(3)] = 3\lambda = 9.$$