Math 490 Test 1, Spring 2018, Solutions

Calculators, books, notes and extra papers are *not* allowed on this test!

Show all work to qualify for full credits

1. (20 points) There are 10 types of coupons. Each time a coupon is collected, it is equally likely to be any of the 10 types. 20 coupons have been collected. Let X be the number of distinct types in the collection. (a) Find $E[X]$; (b) find $\text{Var}(X)$. (Hint: Try to write X as a sum of simpler random variables.)

Solution: For $i = 1, \cdots, 10$, let $X_i = 1$ if there is at least one type i coupon in the collection of 20 coupons and $X_i = 0$ otherwise. Then $X_1 + \cdots + X_{10}$ is the number of distinct types in the collection of 20 coupons. For $i = 1, \cdots, 10$,

$$P(X_i = 0) = \left(\frac{9}{10}\right)^{20}, \quad P(X_i = 1) = 1 - \left(\frac{9}{10}\right)^{20}.$$

For $i \neq j$,

$$P(X_i X_j = 0) = P(X_i = 0) + P(X_j = 0) - P(X_i = 0, X_j = 0) = 2\left(\frac{9}{10}\right)^{20} - \left(\frac{4}{5}\right)^{20},$$

$$P(X_i X_j = 1) = 1 - 2\left(\frac{9}{10}\right)^{20} + \left(\frac{4}{5}\right)^{20}.$$

Thus

$$E[X_i] = 1 - \left(\frac{9}{10}\right)^{20}, \quad \text{Var}(X_i) = \left(\frac{9}{10}\right)^{20}\left(1 - \left(\frac{9}{10}\right)^{20}\right),$$

and

$$\text{Cov}(X_i, X_j) = 1 - 2\left(\frac{9}{10}\right)^{20} + \left(\frac{4}{5}\right)^{20} - \left(1 - \left(\frac{9}{10}\right)^{20}\right)^2.$$

Consequently

$$E[X_1 + \cdots + X_{10}] = 10\left(1 - \left(\frac{9}{10}\right)^{20}\right)$$

and

$$\text{Var}(X_1 + \cdots + X_{10}) = 10\left(\frac{9}{10}\right)^{20}\left(1 - \left(\frac{9}{10}\right)^{20}\right) + 90\left(1 - 2\left(\frac{9}{10}\right)^{20} + \left(\frac{4}{5}\right)^{20} - \left(1 - \left(\frac{9}{10}\right)^{20}\right)^2\right).$$

2. (20 points) The joint density of X and Y is given by

$$f(x,y) = \begin{cases} 6xy(2-x-y) & 0 < x < 1, 0 < y < 1 \\ 0 & \text{otherwise.} \end{cases}$$

(a) For $y \in (0,1)$, find $f_{X|Y}(x|y)$; (b) find $E[X|Y = \frac{1}{2}]$; (c) find $E[X^2|Y = \frac{1}{2}]$.

Solution:

(a) For $y \in (0,1)$ and $x \in (0,1)$,

$$f_{X|Y}(x|y) = \frac{f(x,y)}{f_Y(y)} = \frac{6xy(2-x-y)}{\int_0^1 6xy(2-x-y)dx} = \frac{6x(2-x-y)}{4-3y}.$$

Hence for $y \in (0,1)$,

$$f_{X|Y}(x|y) = \begin{cases} \frac{6x(2-x-y)}{4-3y} & 0 < x < 1 \\ 0 & \text{otherwise,} \end{cases}$$

and

$$f_{X|Y}(x|\frac{1}{2}) = \begin{cases} \frac{6}{5}x(3-2x) & 0 < x < 1 \\ 0 & \text{otherwise.} \end{cases}$$

(b)

$$E[X|Y = \frac{1}{2}] = \int_0^1 x\frac{6}{5}x(3-2x)dx = \frac{3}{5}.$$

(c)

$$E[X^2|Y = \frac{1}{2} = \int_0^1 x^2\frac{6}{5}x(3-2x)dx = \frac{21}{50}.$$

3. (20 points) The number of coins that John picks up when walking to work is a Poisson random variable with mean 2. Each coin is equally likely to be a penny, a nickel, a dime or a quarter. Find the expected amount of money that John picks up on his way to work.

Solution: N be the number of coins that John picks. We are given that N is a Poisson random variable with parameter 2. Let X be the amount of money in cents that John picks up, let A be the number of nickels that Josh picks up, let A be the number of pennies that John picks up, B the number of nickels that John picks up, C the number of dimes that John picks up, D be the number of quarters that John picks up. Then $X = A + 5B + 10C + 25D$. Thus

$$E[X] = E[A + 5B + 10C + 25D] = E[A] + 5E[B] + 10E[C] + 25E[D].$$

$$E[A] = \sum_{n=0}^{\infty} E[A|N = n]P(N = n) = \sum_{n=0}^{\infty} \frac{n}{4}P(N = n)$$
$$= \frac{1}{4}\sum_{n=0}^{\infty} nP(N = n) = \frac{1}{4}E[N] = \frac{1}{2}.$$

Similarly, we also have $E[B] = E[C] = E[D] = \frac{1}{2}$. Thus

$$E[X] = 41 \cdot \frac{1}{2} = \frac{41}{2}.$$

4. (20 points) Consider a Markov chain with states $0, 1, 2, 3, 4, 5$ and transition matrix

$$P = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \frac{1}{3} & 0 & 0 & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

Find the classes, and determine which of the classes are recurrent and which are transient.

Solution:

The classes are $\{0, 1, 2\}$, $\{3\}$ and $\{4, 5\}$. The classes $\{0, 1, 2\}$ and $\{3\}$ are transient, and the class $\{4, 5\}$ is recurrent.