# Solutions to Math 490 Test 3, Spring 2017

Calculators, books, notes and extra papers are *not* allowed on this test!

Show all work to qualify for full credits

1. (20 points) Suppose that $\{N(t), t \geq 0\}$ is a Poisson process with rate $\lambda = 2$. Let $S_n$ be the time of the $n$-th event. Find (a) $E[S_3]$; (b) $\mathrm{Var}(S_3)$; (c) $E[S_5|N(2) = 3]$.

**Solution**. (a) $E[S_3] = \frac{3}{\lambda} = \frac{3}{2}$.
(b) $\mathrm{Var}(S_3) = \frac{3}{\lambda^2} = \frac{3}{4}$.
(c) Given $N(2) = 3$, the conditional distribution of $S_5$ is the same as the unconditional distribution of $2 + S_2$. Thus $E[S_5|N(2) = 3] = 2 + \frac{2}{\lambda} = 3$.

2. (20 points) (a) Let $\{N_1(t), t \geq 0\}$ be a non-homogeneous Poisson process with intensity function $\lambda(t) = 1 + 2t$. Find $\mathrm{Cov}(N_1(2), N_1(3))$.

(b) Suppose that $\{N_2(t), t \geq 0\}$ is a Poisson process with rate $\lambda = 3$ and that $Y_1, Y_2, \dots$ are independent uniform random variables on $(0, 2)$. Assume that $Y_1, Y_2, \dots$ are also independent of $\{N_2(t), t \geq 0\}$. Define

$$X(t) = \sum_{i=1}^{N_2(t)} Y_i, \qquad t \geq 0.$$

Find $\mathrm{Cov}(X(2), X(3))$.

**Solution**. (a) By the independent increments property,

$$\begin{aligned}&\mathrm{Cov}(N_1(2), N_1(3)) = \mathrm{Cov}(N_1(2), N_1(2) + (N_1(3) - N_1(2))\\&= \mathrm{Cov}(N_1(2), N_1(2)) = \mathrm{Var}(N_1(2)).\end{aligned}$$

Since $N_1(2)$ is a Poisson random variable with parameter

$$\int_0^2 (1 + 2t)dt = 6,$$

we get that $\mathrm{Cov}(N_1(2), N_1(3)) = 6$.

(b) By the independent increments property,

$$\begin{aligned}&\mathrm{Cov}(X(2), X(3)) = \mathrm{Cov}(X(2), X(2) + (X(3) - X(2))\\&= \mathrm{Cov}(X(2), X(2)) = \mathrm{Var}(X(2)) = 6E[Y_1^2] = 8.\end{aligned}$$

3. (20 points) Consider a job shop that consists of 5 machines and one serviceman. Suppose that the amount of time that each machine runs before breaking down is an exponential random variable with parameter 2, and suppose that the amount of time it takes the serviceman to fix a machine is an exponential random variable with parameter 4. Suppose that there are always jobs for the machines to do and that the serviceman keeps on working as long as there are repair work to be done. For $t \geq 0$, let $X(t)$ be the number of machines that are not

in use (either being fixed or waiting to be fixed) at time $t$. $\{X(t), t \geq 0\}$ is a continuous-time Markov chain. Write down the birth and death rates for this continuous-time Markov chain.

**Solution**. This is basically Example 6.13 in the book. $\mu_0 = 0, \mu_n = 4$, $n = 1, \ldots, 5$. $\lambda_0 = 10, \lambda_1 = 8, \lambda_2 = 6, \lambda_3 = 4, \lambda_4 = 2, \lambda_5 = 0$.

4. (20 points) Suppose that $\{X(t), t \geq 0\}$ is a Yule process, that is, a pure birth process with birth rates $\lambda_n = n\lambda$, $n \geq 0$. Find the expected time for the process to go from 1 to 3.

**Solution**.

$$E[\text{ expected to go from 1 to 3}] = E[T_1] + E[T_2] = \frac{1}{\lambda} + \frac{1}{2\lambda}.$$