Solutions to Math 490 Test 2, Spring 2017

Calculators, books, notes and extra papers are *not* allowed on this test!

Show all work to qualify for full credits

1. (20 points) Let X_n be a Markov chain with states 0, 1, 2 and transition matrix

$$P = \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \end{pmatrix}.$$

Find the long-run proportions for this Markov chain.

Solution By solving the system

$$\begin{aligned}
\pi_0 &= \frac{1}{4}\pi_1 + \frac{1}{4}\pi_2 \\
\pi_1 &= \frac{1}{2}\pi_0 + \frac{1}{2}\pi_1 + \frac{1}{4}\pi_2 \\
\pi_2 &= \frac{1}{2}\pi_0 + \frac{1}{4}\pi_1 + \frac{1}{2}\pi_2 \\
\pi_0 &+ \pi_1 + \pi_2 = 1,
\end{aligned}$$

we get that $\pi_0 = \frac{1}{5}, \pi_1 = \frac{2}{5}, \pi_2 = \frac{2}{5}$.

2. (20 points) Consider a branching process X_n with offspring distribution given by $P_0 = 1/4$, $P_1 = 1/4$ and $P_2 = 1/2$. Find the extinction probability π_0 given that $X_0 = 1$.

By solving the equation

$$\pi_0 = \frac{1}{4} + \frac{1}{4}\pi_0 + \frac{1}{2}\pi_0^2,$$

we get two solutions: 1/2 and 1. Thus $\pi_0 = 1/2$.

3. (20 points) Consider a post office with 2 clerks. Suppose that when John enters the post office he finds that David is being served by one of the clerks and Mike by the other. John is told that he will be served as soon as either David or Mike is done. If the amount of time that a clerk spends with a customer is exponentially distributed with parameter $\lambda = 1/2$. Find the probability that John is not the last to leave the post office. We assume that each customer leaves the post office as soon as his/her service is finished.

Solution Let A be the event that David finishes before Mike and let E be the event that John is not the last to leave the post office. Then, by conditioning, we have

$$P(E) = P(E|A)P(A) + P(E|A^c)P(A^c) = P(E|A)\frac{1}{2} + P(E|A^c)\frac{1}{2}.$$

By using the memoryless property of exponential random variables, we get

$$P(E|A) = \frac{1}{2}, \qquad P(E|A^c) = \frac{1}{2}.$$

Thus

$$P(E) = \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 = \frac{1}{2}.$$

4. (20 points) Let X and Y be independent exponential random variables with parameters $\lambda = 1$ and $\mu = 2$ respectively. Let $M = \min(X, Y)$. Find $E[MX|M = Y]$.

Solution By the memoryless propertyod exponential random variables, given $M = Y$, X is distributed as $M + X'$, where X' is an exponential random variable with parameter $\lambda = 1$ and independent of M. Thus

$$\begin{aligned} E[MX|M = Y] &= E[M(M + X')] \\ &= E[M^2] + E[M]E[X'] \\ &= \frac{2}{(\lambda+\mu)^2} + \frac{1}{\lambda(\lambda+\mu)} = \frac{5}{9}. \end{aligned}$$