Solutions to Math 490 Test 1, Spring 2017

Calculators, books, notes and extra papers are not allowed on this test!

Show all work to qualify for full credits

1. (20 points) An urn contains 10 orange balls labeled 1, 2, . . . , 10, and 20 blue balls labeled 1, 2, . . . , 20. 8 balls are randomly selected from the urn without replacement. (a) For \( i = 1, 2, . . . , 10 \), find the probability that the \( i \)-th orange ball is among the selected; (b) for \( i, j = 1, 2, . . . , 10 \), \( i \neq j \), find the probability that the \( i \)-th and \( j \)-th orange balls are both among the selected; (c) let \( X \) be the number of orange balls among the selected, find \( E[X] \); (d) find \( \text{Var}(X) \).

**Solution**

(a) For \( i = 1, 2, . . . , 10 \), let \( A_i \) be the event that the \( i \)-th orange ball is among the selected. Then

\[
P(A_i) = \frac{\binom{29}{7}}{\binom{30}{8}} = \frac{8}{30}.
\]

(b) For \( i, j = 1, 2, . . . , 10 \), \( i \neq j \),

\[
P(A_i \cap A_j) = \frac{\binom{28}{6}}{\binom{30}{8}} = \frac{7 \cdot 8}{29 \cdot 30}.
\]

(c) For \( i = 1, 2, . . . , 10 \), let \( X_i = 1 \) if the \( i \)-th orange ball is among the selected and 0 otherwise. Then \( X = X_1 + \cdots + X_{10} \). Hence

\[
E[X] = E[X_1] + \cdots + E[X_{10}] = \frac{80}{30} = \frac{8}{3}.
\]

(d) For \( i = 1, \ldots, 10 \), \( \text{Var}(X_i) = \frac{8}{30} \cdot \frac{22}{30} \). For \( i, j = 1, 2, \ldots, 10 \), \( i \neq j \),

\[
\text{Cov}(X_i, X_j) = E[X_i X_j] - E[X_i]E[X_j] = \frac{7 \cdot 8}{29 \cdot 30} - \frac{8}{30} \cdot \frac{8}{30}.
\]

Thus

\[
\text{Var}(X) = \sum_{i=1}^{10} \text{Var}(X_i) + \sum_{i \neq j} \text{Cov}(X_i, X_j)
\]

\[
= 10 \cdot \frac{8}{30} \cdot \frac{22}{30} + 90 \cdot \left( \frac{7 \cdot 8}{29 \cdot 30} - \frac{8}{30} \cdot \frac{8}{30} \right).
\]

2. (20 points) The joint density of \( X \) and \( Y \) is given by

\[
f(x, y) = \begin{cases} 
\frac{2}{y} e^{-2y} & 0 < x < y, 0 < y < \infty \\
0 & \text{otherwise.}
\end{cases}
\]
For \( y > 0 \), (a) find \( f_{X|Y}(x|y) \); (b) find \( E[X|Y = y] \); (c) Find \( E[X^2|Y = y] \).

**Solution** (a)
\[
f_Y(y) = \int_0^y \frac{2}{y} e^{-2y} dx = 2e^{-2y}
\]
and
\[
f_{X|Y}(x|y) = \begin{cases} \frac{1}{y}, & x \in (0, y) \\ 0, & \text{otherwise.} \end{cases}
\]

(b)
\[
E[X|Y = y] = \int_0^y \frac{1}{y} x dx = \frac{y^2}{2}.
\]

(c)
\[
E[X^2|Y = y] = \int_0^y \frac{1}{y} x^2 dx = \frac{y^2}{3}.
\]

3. (20 points) An essay is sent to an office consisting of two typists, A and B. If it is typed by A, the number of typos is a Poisson random variable with parameter 2. If it is typed by B, the number of typos is a Poisson random variable with parameter 3. Let \( X \) be the number of typos in the typed essay. Assume that each typist is equally like to do the work. (a) Find \( E[X] \); (b) find \( \text{Var}(X) \).

**Solution** Let \( Y = 1 \) if A is chosen and \( Y = 2 \) if B is chosen. Then

(a)
\[
E[X] = \frac{1}{2} (E[X|Y = 1] + E[X|Y = 2])
= \frac{1}{2} (2 + 3) = 2.5.
\]

(b)
\[
E[X^2] = \frac{1}{2} (E[X^2|Y = 1] + E[X^2|Y = 2])
= \frac{1}{2} (2^2 + 2^2 + 3 + 3^2) = 9
\]
and so
\[
\text{Var}(X) = E[X^2] - (E[X])^2 = 9 - \frac{25}{4}.
\]

4. (20 points) Consider a Markov chain with states 0, 1, 2, 3, 4 and transition matrix
\[
P = \begin{pmatrix}
1 & \frac{1}{2} & 0 & 0 & 0 \\
\frac{1}{2} & 1 & \frac{1}{2} & 0 & 0 \\
0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 \\
0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 \\
\frac{1}{4} & \frac{1}{4} & 0 & 0 & \frac{1}{2}
\end{pmatrix}.
\]
Find the classes, and determine which of the classes are recurrent and which are transient.

**Solution** The classes are $\{0, 1\}$, $\{2, 3\}$ and $\{4\}$. The classes $\{0, 1\}$ and $\{2, 3\}$ are recurrent, and the class $\{4\}$ is transient.