Solutions to Math 490 Test 1, Spring 2017

Calculators, books, notes and extra papers are *not* allowed on this test!

Show all work to qualify for full credits

1. (20 points) An urn contains 10 orange balls labeled $1, 2, \ldots, 10$, and 20 blue balls labeled $1, 2, \ldots, 20$. 8 balls are randomly selected from the urn without replacement. (a) For $i = 1, 2, \ldots, 10$, find the probability that the i-th orange ball is among the selected; (b) for $i, j = 1, 2, \ldots, 10$, $i \neq j$, find the probability that the i-th and j-th orange balls are both among the selected; (c) let X be the number of orange balls among the selected, find $E[X]$; (d) find $\mathrm{Var}(X)$.

Solution (a) For $i = 1, 2, \ldots, 10$, let A_i be the event that the i-th orange ball is among the selected. Then

$$P(A_i) = \frac{\binom{29}{7}}{\binom{30}{8}} = \frac{8}{30}.$$

(b) For $i, j = 1, 2, \ldots, 10$, $i \neq j$,

$$P(A_i \cap A_j) = \frac{\binom{28}{6}}{\binom{30}{8}} = \frac{7 \cdot 8}{29 \cdot 30}.$$

(c) For $i = 1, 2, \ldots, 10$, let $X_i = 1$ if the i-th orange ball is among the selected and 0 otherwise. Then $X = X_1 + \cdots + X_{10}$. Hence

$$E[X] = E[X_1] + \cdots + E[X_{10}] = \frac{80}{30} = \frac{8}{3}.$$

(d) For $i = 1, \ldots, 10$, $\mathrm{Var}(X_i) = \frac{8}{30} \cdot \frac{22}{30}$. For $i, j = 1, 2, \ldots, 10$, $i \neq j$,

$$\mathrm{Cov}(X_i, X_j) = E[X_i X_j] - E[X_i]E[X_j] = \frac{7 \cdot 8}{29 \cdot 30} - \frac{8}{30} \cdot \frac{30}{30}.$$

Thus

$$\begin{aligned} \mathrm{Var}(X) &= \sum_{i=1}^{10} \mathrm{Var}(X_i) + \sum_{i \neq j} \mathrm{Cov}(X_i, X_j) \\ &= 10 \cdot \frac{8}{30} \cdot \frac{22}{30} + 90 \cdot \left(\frac{7 \cdot 8}{29 \cdot 30} - \frac{8}{30} \cdot \frac{8}{30} \right). \end{aligned}$$

2. (20 points) The joint density of X and Y is given by

$$f(x, y) = \begin{cases} \frac{2}{y} e^{-2y} & 0 < x < y, 0 < y < \infty \\ 0 & \text{otherwise.} \end{cases}$$

For $y > 0$, (a) find $f_{X|Y}(x|y)$; (b) find $E[X|Y = y]$; (c) Find $E[X^2|Y = y]$.

Solution (a)

$$f_Y(y) = \int_0^y \frac{2}{y} e^{-2y} dx = 2e^{-2y}$$

and

$$f_{X|Y}(x|y) = \begin{cases} \frac{1}{y}, & x \in (0, y) \\ 0, & \text{otherwise.} \end{cases}$$

(b)

$$E[X|Y = y] = \int_0^y x \frac{1}{y} dx = \frac{y}{2}.$$

(c)

$$E[X^2|Y = y] = \int_0^y x^2 \frac{1}{y} dx = \frac{y^2}{3}.$$

3. (20 points) An essay is sent to an office consisting of two typists, A and B. If it is typed by A, the number of typos is a Poisson random variable with parameter 2. If it is typed by B, the number of typos is a Poisson random variable with parameter 3. Let X be the number of typos in the typed essay. Assume that each typist is equally like to do the work. (a) Find $E[X]$; (b) find $\text{Var}(X)$.

Solution Let $Y = 1$ if A is chosen and $Y = 2$ if B is chosen. Then
(a)

$$\begin{aligned} E[X] &= \frac{1}{2} \left(E[X|Y = 1] + E[X|Y = 2] \right) \\ &= \frac{1}{2}(2 + 3) = 2.5. \end{aligned}$$

(b)

$$\begin{aligned} E[X^2] &= \frac{1}{2} \left(E[X^2|Y = 1] + E[X^2|Y = 2] \right) \\ &= \frac{1}{2}(2 + 2^2 + 3 + 3^2) = 9 \end{aligned}$$

and so

$$\text{Var}(X) = E[X^2] - (E[X])^2 = 9 - \frac{25}{4}.$$

4. (20 points) Consider a Markov chain with states $0, 1, 2, 3, 4$ and transition matrix

$$P = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{4} & \frac{1}{4} & 0 & 0 & \frac{1}{2} \end{pmatrix}.$$

Find the classes, and determine which of the classes are recurrent and which are transient.

Solution The classes are $\{0, 1\}$, $\{2, 3\}$ and $\{4\}$. The classes $\{0, 1\}$ and $\{2, 3\}$ are recurrent, and the class $\{4\}$ is transient.