1. (due 09/06) If 10 married couples are seated at random at a round table, find the probability that no husband sits next to his wife.

2. (due 09/06) A hand of 13 cards is chosen randomly from an ordinary deck of 52 cards. Find the probability that
(a) the hand contains the ace and king of some suit;
(b) the hand contains all 4 of at least 1 of the 13 denominations.
(c) at least one suit is missing from the hand.

3. (due 09/13) In successive rolls of a pair of fair dice, find the probability of getting 2 sevens before 6 even numbers.

4. (due 09/13) Two gamblers, A and B, bet on the outcomes of successive flips of a coin. On each flip, if the coin comes up heads, A collects 1 unit from B, whereas if it comes up tails, A pays 1 unit to B. They continue to do so until one of them runs out of money. If it is assumed that the successive flips of the coin are independent and each flip results in a success with probability $p$, what is the probability that A ends up with all the money if he starts with $i$ units and B starts with $N - i$ units?

5. (due 09/13) A round-robin tournament of $n$ contestants is one in which each pair of contestants play each other exactly once, with the outcome of any play being that one of the contestants wins and the other loses. For a fixed integer $k$, $k < n$, a question of interest is whether it is possible that the tournament outcome is such that for every set of $k$ players there is a player who beat each member of this set. Show that if

$$\binom{n}{k} (1 - 2^{-k})^{n-k} < 1$$

then there is such a tournament.

6. (due 09/20) Let $X$ be uniformly distributed on $(0, 1)$. Find the density of $Y = -\lambda^{-1} \ln(1 - X)$.

7. (due 09/20) Let $X$ be an absolutely continuous random variable with density $f$. Find the density of $Y = X^2$.

8. (due 09/20) Let $X$ and $Y$ be independent geometric random variables with parameter $p$. (a) Find the distribution $\min(X, Y)$; (b) Find $P(Y \geq X)$; (c) Find the distribution of $X + Y$; and (d) Find $P(Y = y | X + Y = z)$ for $z > 0$ and $y = 1, 2, \ldots, z - 1$.

9. (due 09/27) Suppose that there are $N$ different types of coupons and each time one
obtains a coupon it is equally likely to be any one of the $N$ types. Let $S_n$ be the number of different types of coupons that are contained in a set of $n$ coupons. Find $ES_n$.

10. (due 09/27) Suppose that $n$ balls are distributed at random into $r$ boxes. Let $X$ be the number of empty boxes. Find $EX$ and Var$X$.

11. (due 09/27) Suppose that a deck of $n$ cards contain $n_i$ cards of type $i$ for $i = 1, \ldots, r$, $\sum_{i=1}^{r} n_i = n$. The deck is well shuffled so that all $n!$ orderings of the cards are equally likely, and the cards are then turned over one at a time. Find the expected number of cards that need to be turned over until all types have appeared at least once.

12. (due 09/27) For $2 \leq i < n$, show that there is a 2-coloring of $K_n$, the complete graph on $n$ vertices, with at most

$$\binom{n}{i} \left( 1 - \binom{i}{2} \right)$$

monochromatic $K_i$.

13. (due 10/07) Independent trials, each result in a success with probability $p \in (0, 1)$ and a failure with probability $1 - p$, are performed. Show that with probability 1 the pattern $SFSF$ appear infinitely often. (Hint: Borel-Cantelli lemmas.)

14. (due 10/07) Let $X_1, X_2, \ldots$ be independent with $P(X_n = 1) = p_n$ and $P(X_n = 0) = 1 - p_n$. Show that $X_n \to 0$ almost surely if and only if $\sum_n p_n < \infty$. (Hint: Borel-Cantelli lemmas.)

15. (due 10/07) Let $f$ be a continuous function on $[0, 1]$, and let

$$f_n(x) = \sum_{m=0}^{n} \binom{n}{m} x^m (1-x)^{n-m} f\left( \frac{m}{n} \right)$$

be the Bernstein Polynomial of degree $n$ associated with $f$. Using the Weak Law to show that as $n \to \infty$,

$$\sup_{x \in [0,1]} |f_n(x) - f(x)| \to 0.$$ 

16. (due 10/07) Let $X_1, X_2, \ldots$ be iid with $0 < X_i < \infty$. Let $T_n = X_1 + \cdots + X_n$ and think of $T_n$ as the time of the $n$-th occurrence of some event. For a concrete situation consider a diligent janitor who replaces a lightbulb the instant it burns out. Suppose that the first bulb is put in at time 0 and let $X_i$ be the lifetime of the $i$-th lightbulb. In this interpretation $T_n$ is the time the $n$-th lightbulb burns out and $N_i = \sup \{n : T_n \leq t \}$ is the number of lightbulbs that have burned out by time $t$. Show that, if $EX_1 = \mu < \infty$, then

$$\frac{N_i}{t} \to \frac{1}{\mu}, \quad t \to \infty.$$ 

2
17. (due 10/21) Suppose that $X_n \Rightarrow X$ and $Y_n \Rightarrow c$, where $c$ is a constant. Show that $X_n + Y_n \Rightarrow X + c$.

18. (due 10/21) Suppose that $X_n \Rightarrow X$, $Y_n \geq 0$ and $Y_n \Rightarrow c$, where $c > 0$ is a positive constant. Show that $X_n Y_n \Rightarrow cX$. (This result is true without the assumptions $Y_n \geq 0$ and $c > 0$.)

19. (due 10/21) Let $X_1, X_2, \ldots$ be independent and identically distributed with $EX_1 = 0$ and $E(X_1^2) = \sigma^2 \in (0, \infty)$. Show that

$$\frac{\sum_{m=1}^{n} X_m}{\sqrt{\sum_{m=1}^{n} X_m^2}}$$

converges weakly to a standard normal random variable.

20. (due 11/11) Let $\xi_1, \xi_2, \ldots$ be independent and identically distributed random variables with $P(\xi_1 = 1) = p \in (0, 1)$ and $P(\xi_1 = -1) = 1 - p$. Put $S_0 = 0$ and $S_n = \xi_1 + \cdots + \xi_n$.

(a) Suppose $p > 1/2$ and let $\phi(x) = ((1 - p)/p)^x$. Show that $\phi(S_n)$ is a martingale.

(b) If $T_x = \inf\{n : S_n = x\}$, then for $a < 0 < b$,

$$P(T_a < T_b) = \frac{\phi(b) - \phi(0)}{\phi(b) - \phi(a)}.$$  

(c) If $a < 0$, then $P(\min_n S_n \leq a) = P(T_a < \infty) = ((1 - p)/p)^{-a}$. If $b > 0$, then $P(T_b < \infty) = 1$.

(d) If $b > 0$, then $ET_b = b/(2p - 1)$.

(e) Let $p > 1/2$ and $\sigma^2 = 1 - (2p - 1)^2$. Show that $X_n = (S_n - (2p - 1)n)^2 - \sigma^2n$ is a martingale and that $\text{Var}(T_1) = \sigma^2/(2p - 1)^3$.

(f) Let $p = 1/2$ and $T = \inf\{n : S_n \notin (-a, a)\}$. Show that $S_n^2 - n$ is a martingale and that $ET = a^2$.

(g) Let $p = 1/2$. Find constants $b$ and $c$ such that $Y_n = S_n^4 - 6nS_n^2 + bn^2 + cn$ is a martingale and use this to find $E(T^2)$.

21 (due 12/04) Let $X = (X_1, \ldots, X_n)$ be a family of independent random variables with $X_k$ taking values in $\Omega_k$, and let $\Omega = \prod \Omega_k$. Suppose that $f$ is a real valued function $\Omega$ such that, for each $x \in \Omega$, there exists a nonnegative unit $n$-vector $\alpha$ such that

$$f(x) \leq f(y) + c d\alpha(x, y), \quad \forall y \in \Omega. \quad (1)$$

Show that

$$P(|f(X) - m| \geq t) \leq 4e^{-t^2/(4\varsigma^2)},$$

3
where $m$ is the median of $f(X)$. Show that the same conclusion holds if (1) is replaced by

$$f(y) \leq f(x) + cd_a(x, y), \quad \forall y \in \Omega.$$  

(Hint: Use Talagrand’s inequality.)

22 (due 12/04) A random knight makes each permissible move with equal probability. If it starts in a corner, how long on average will it take to return to the starting point?