9th Homework Set — Solutions

Chapter 6

Problem 6.48 Let $X_1, \ldots, X_5$ be independent exponential random variables with parameter λ.

(a)

$$\begin{aligned}
P(\min(X_1, \ldots, X_5) \le a) &= 1 - P(\min(X_1, \ldots, X_5) > a) \\
&= 1 - P(X_1 > a, \ldots, X_5 > a) \\
&= 1 - P(X_1 > a) \cdots P(X_5 > a) \\
&= \begin{cases} 1 - (e^{-\lambda a})^5 & a > 0 \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}$$

(b)

$$\begin{aligned}
P(\max(X_1, \ldots, X_5) \le a) &= P(X_1 \le a, \ldots, X_5 \le a) \\
&= P(X_1 \le a) \cdots P(X_5 \le a) \\
&= \begin{cases} (1 - e^{-\lambda a})^5 & a > 0 \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}$$

Chapter 7

Problem 7.5 If (X, Y) is the location of the accident, then X and Y are uniform random variables on $(-\frac{3}{2}, \frac{3}{2})$. Let $D = |X| + |Y|$. Then

$$\begin{aligned}
E[D] &= E[|X|] + E[|Y|] = 2E[|X|] \\
&= 2\int_{-\frac{3}{2}}^{\frac{3}{2}} \frac{|x|}{3} dx = \frac{4}{3}\int_0^{\frac{3}{2}} x dx \\
&= \frac{4}{3} \cdot \frac{9}{8} = \frac{3}{2}.
\end{aligned}$$

Problem 7.6 Let X_i be the outcome of the i-th roll of the die, for $i = 1, \ldots, 10$, and note that $E[X_i] = \frac{7}{2}$. Let $X = X_1 + \cdots + X_{10}$. Now $E[X] = E[X_1] + \cdots + E[X_{10}] = 10E[X_1] = 35$.

Problem 7.7 (a) Let X_i be one if both A and B choose the i-th object, for $i = 1, \ldots, 10$. Then $E[X_i] = P(X_i = 1) = \left(\frac{3}{10}\right)^2 = \frac{9}{100}$. Now, the expected number of objects chosen by both A and B is $E[X_1] + \cdots + E[X_{10}] = 10E[X_1] = 0.9$.

(b) Let Y_i be one if neither A nor B choose the i-th object. Then $E[Y_i] = P(Y_i = 1) = \left(\frac{7}{10}\right)^2 = \frac{49}{100}$, so that $E[Y_1 + \cdots + Y_{10}] = 10E[Y_1] = 4.9$.

(c) Let Z_i be one if either A or B (but not both) chooses the i-th object. Then $E[Z_i] = P(Z_i = 1) = 2\frac{3}{10}\frac{7}{10} = \frac{21}{50}$. Now, $E[Z_1 + \cdots + Z_{10}] = 10E[Z_1] = \frac{21}{5} = 4.2$.

Problem 7.8 Following the hint, let X_1 be one if the i-th arrival sits at a previously unoccupied table. Then $E[X_i] = P(X_i = 1) = (1-p)^{i-1}$, so that

$$E[X_1 + \cdots + X_N] = \sum_{i=1}^{N} (1-p)^{i-1} = \frac{1-(1-p)^N}{1-(1-p)} = \frac{1-(1-p)^N}{p}.$$

Problem 7.11 Let X_i be one if the i-th outcome differs from the $(i-1)$-th outcome, for $i = 2, \ldots, n$. We have $E[X_i] = P(X_i = 1) = 2p(1-p)$, so that $E[X_2 + \cdots + X_n] = 2(n-1)p(1-p)$.

Problem 7.18 Let X_i be one if the i-th card is a match, for $i = 1, \ldots, 52$, and let $X = X_1 + \cdots + X_{52}$. Then $P(X_i = 1) = \frac{1}{13}$, so that $E[X] = 52E[X_1] = \frac{52}{13} = 4$.

Problem 7.19 (a) If X is the number of insects caught before a type 1 catch, then $(X+1)$ is geometric with parameter P_1, so that $E[X] = \frac{1}{P_1} - 1$.

(b) Let Y_i be one if an insect of type i is caught before an insect of type 1, for $i = 2, \ldots, r$. Then $Y = Y_2 + \cdots + Y_r$ is the number of insects caught before an insect of type 1. We have $E[Y_i] = P(Y_i = 1) = \frac{P_i}{P_i + P_1}$, so that

$$E[Y] = \sum_{i=2}^{r} \frac{P_i}{P_i + P_1}.$$

Problem 7.21 (a) Let X be the number of days of the year that are birthdays of exactly 3 people. For $i = 1, \ldots, 365$, let $X_i = 1$ if the i-day is the birthday of eaxctly 3 people and $X_i = 0$ otherwise. Then $X = \sum_{i=1}^{365} X_i$. Since for each i,

$$EX_i = P(X_i = 1) = \binom{100}{3} \left(\frac{1}{365}\right)^3 \left(\frac{364}{365}\right)^{97},$$

we get that

$$EX = 365 \binom{100}{3} \left(\frac{1}{365}\right)^3 \left(\frac{364}{365}\right)^{97}.$$

(b) Let Y be the number of distinct birthdays. For $i = 1, \ldots, 365$, let $Y_i = 1$ if the i-day is someone's birthday and $Y_i = 0$ otherwise. Then $Y = \sum_{i=1}^{365} Y_i$. Since for each i,

$$EY_i = P(Y_i = 1) = 1 - P(Y_i = 0) = 1 - \left(\frac{364}{365}\right)^{100},$$

we get that

$$EY = 365 \left[1 - \left(\frac{364}{365}\right)^{100}\right].$$

Problem 7.30 Note that $E[X^2] = E[Y^2] = \mathrm{Var}(X) + E[X]^2 = \sigma^2 + \mu^2$. Now we conclude that

$$E\left[(X-Y)^2\right] = E\left[X^2\right] - 2E[X]E[Y] + E\left[Y^2\right] = 2\sigma^2,$$

using the fact that X and Y are independent.

Problem 7.31 Let X_i be the outcome of the i-th roll of the die, for $i = 1, \ldots, 10$. Then $\mathrm{Var}(X_i) = \frac{35}{12}$, so that

$$\mathrm{Var}(X_1 + \cdots + X_{10}) = 10 \cdot \frac{35}{12} = \frac{175}{6}.$$

Problem 7.33 (a)

$$E\left[(2+X)^2\right] = 4 + 4E[X] + E\left[X^2\right] = 8 + \mathrm{Var}(X) + E[X]^2 = 14.$$

(b)

$$\mathrm{Var}(4 + 3X) = 9\mathrm{Var}(X) = 45.$$

Problem 7.38 We have

$$\begin{aligned}
E\left[XY\right] &= \int_0^\infty \int_0^x 2ye^{-2x} dydx = \int_0^\infty x^2 e^{-2x} dx = \frac{1}{4}, \\
E\left[X\right] &= \int_0^\infty \int_0^x 2e^{-2x} dydx = \int_0^\infty 2xe^{-2x} dx = \frac{1}{2}, \quad \text{and} \\
E\left[Y\right] &= \int_0^\infty \int_0^x \frac{2y}{x} e^{-2x} dydx = \int_0^\infty xe^{-2x} dx = \frac{1}{4}.
\end{aligned}$$

Hence,

$$\mathrm{Cov}\left(X, Y\right) = \frac{1}{4} - \frac{1}{2} \cdot \frac{1}{4} = \frac{1}{8}.$$

Problem 7.39 We have

$$\begin{aligned}
\mathrm{Cov}\left(Y_n, Y_n\right) &= \mathrm{Var}\left(Y_n\right) = 3\sigma^2, \\
\mathrm{Cov}\left(Y_n, Y_{n+1}\right) &= \mathrm{Cov}\left(X_n + X_{n+1} + X_{n+2}, X_{n+1} + X_{n+2} + X_{n+3}\right) \\
&= \mathrm{Cov}\left(X_{n+1} + X_{n+2}, X_{n+1} + X_{n+2}\right) \\
&= \mathrm{Var}\left(X_{n+1} + X_{n+2}\right) = 2\sigma^2, \\
\mathrm{Cov}\left(Y_n, Y_{n+2}\right) &= \mathrm{Cov}\left(X_n + X_{n+1} + X_{n+2}, X_{n+2} + X_{n+3} + X_{n+4}\right) \\
&= \mathrm{Cov}\left(X_{n+2}, X_{n+2}\right) = \mathrm{Var}\left(X_{n+2}\right) = \sigma^2, \quad \text{and} \\
\mathrm{Cov}\left(Y_n, Y_{n+j}\right) &= 0 \quad \text{if } j \geq 3.
\end{aligned}$$

Problem 7.41 the number of carp is a hypergeometric random variable, so that we have

$$E\left[X\right] = \frac{20 \cdot 30}{100} = 6,$$

and

$$\mathrm{Var}\left(X\right) = \frac{20 \cdot 80}{99} \cdot \frac{3}{10} \cdot \frac{7}{10} = \frac{112}{33}.$$

Problem 7.42 (a) Let X_i be one if the i-th pair consists of a man and a women, and zero otherwise. Then the sum $X_1 + \cdots + X_{10}$ is the number of pairs that consist of a man and a woman.
We have $E\left[X_i\right] = P(X_i = 1) = 2 \cdot \frac{10 \cdot 10}{20 \cdot 19} = \frac{10}{19}$, so that

$$E\left[X_1 + \cdots + X_{10}\right] = \frac{100}{19}.$$

Now, we have $\text{Var}(X_i) = E[X_i^2] - E[X_i]^2 = \frac{10}{19} - \frac{100}{361} = \frac{90}{361}$, and $\text{Cov}(X_i, X_j) = E[X_iX_j] - E[X_i]E[X_j] = \frac{10}{19} \cdot \frac{9}{17} - \frac{100}{361} = \frac{10}{6137}$ if $i \neq j$, so that

$$\text{Var}(X_1 + \cdots + X_{10}) = \frac{900}{361} + 10 \cdot 9 \cdot \frac{10}{6137} = \frac{16200}{6137} = 2.6397.$$

(b) Let Y_i be one if the i-th couple are paired together. $E[Y_i] = P(Y_i = 1) = \frac{2 \cdot 10 \cdot 18!}{20!} = \frac{1}{19}$, so that

$$E[Y_1 + \cdots + Y_{10}] = \frac{10}{19}.$$

We have $\text{Var}(Y_i) = E[Y_i^2] - E[Y_i]^2 = \frac{1}{19} - \frac{1}{361} = \frac{18}{361}$ and $E[Y_iY_j] = \frac{8\binom{10}{2} \cdot 16!}{20!} = \frac{1}{323}$, so that $\text{Cov}(Y_i, Y_j) = \frac{1}{323} - \frac{1}{361} = \frac{2}{6137}$, so that

$$E[Y_1 + \cdots + Y_{10}] = \frac{10}{19}.$$

We have $\text{Var}(Y_i) = E[Y_i^2] - E[Y_i]^2 = \frac{1}{19} - \frac{1}{361} = \frac{18}{361}$ and $E[Y_iY_j] = \frac{8\binom{10}{2} \cdot 16!}{20!} = \frac{1}{323}$, so that

$$\text{Var}(Y_1 + \cdots + Y_{10}) = \frac{180}{361} + 90 \cdot \frac{2}{6137} = \frac{3240}{6137}.$$