

Fifth Homework Set — Solutions

Chapter 4

Problem 55

$$\begin{aligned}
 P(\text{no errors}) &= P(\text{no errors}|\text{first typist})P(\text{first typist}) \\
 &\quad + P(\text{no errors}|\text{second typist})P(\text{second typist}) \\
 &= \frac{1}{2} \left(\frac{3^0}{0!}e^{-3} + \frac{4 \cdot 2^0}{0!}e^{-4.2} \right) \\
 &= \frac{1}{2} (e^{-3} + e^{-4.2}).
 \end{aligned}$$

Problem 57 X is Poisson with parameter $\lambda = 3$.

$$\begin{aligned}
 \text{(a)} \quad P(X \geq 3) &= 1 - P(X = 0) - P(X = 1) - P(X = 2) = 1 - e^{-3} \left(1 + 3 + \frac{9}{2} \right) = 0.5768. \\
 \text{(b)} \quad P(X \geq 3|X \geq 1) &= \frac{P(X \geq 3)}{P(X \geq 1)} = \frac{P(X \geq 3)}{1 - e^{-3}} = 0.6070.
 \end{aligned}$$

Problem 59 Let X be the number of times you win a prize. Then X is binomial with $n = 50$ and $p = \frac{1}{100}$, i.e., we can use the Poisson approximation with $\lambda = 50 \cdot \frac{1}{100} = \frac{1}{2}$.

$$\begin{aligned}
 \text{(a)} \quad P(X \geq 1) &= 1 - P(X = 0) = 1 - e^{-\frac{1}{2}} = 0.3935 \\
 \text{(b)} \quad P(X = 1) &= \frac{1}{2}e^{-\frac{1}{2}} = 0.3033 \\
 \text{(c)} \quad P(X \geq 2) &= 1 - P(X = 0) - P(X = 1) = 1 - e^{-\frac{1}{2}} \left(1 + \frac{1}{2} \right) = 0.0902
 \end{aligned}$$

Problem 61 Let X be Poisson with parameter $\lambda = 1000 \cdot 0.0014 = 1.4$. Then $P(X \geq 2) = 1 - P(X = 0) - P(X = 1) = 1 - e^{-1.4}(1 + 1.4) = 0.4082$.

Problem 63 Let X be a Poisson random variable with parameter $\lambda = \frac{5}{2}$. Then X gives a reasonable description of the number of people entering the casino between 12 and 12:05.

$$\begin{aligned}
 \text{(a)} \quad P(X = 0) &= e^{-\frac{5}{2}} = 0.0821 \\
 \text{(b)} \quad P(X \geq 4) &= 1 - e^{-\frac{5}{2}} \left(1 + \frac{5}{2} + \frac{25}{8} + \frac{125}{48} \right) = 0.2424
 \end{aligned}$$

Problem 72 Let A be the stronger team. $P(A \text{ wins in } i \text{ games}) = \binom{i-1}{i-4} 0.6^i 0.4^{i-4}$, for $i = 4, \dots, 7$. Hence

$$P(A \text{ wins best-of-seven series}) = \sum_{i=4}^7 \binom{i-1}{i-4} 0.6^4 0.4^{i-4} = 0.7102.$$

Similarly,

$$P(A \text{ wins best-of-three series}) = \sum_{i=2}^3 \binom{i-1}{i-2} 0.6^4 0.4^{i-2} = 0.6480.$$

Problem 73 Let X be the number of games played in a match. Then $P\{X = i\} = 2 \binom{i-1}{i-4} \left(\frac{1}{2}\right)^i$ for $i = 4, \dots, 7$. Hence, $E[X] = 2 \sum_{i=4}^7 i \binom{i-1}{i-4} \left(\frac{1}{2}\right)^i = 5.8125$.

Problem 77 Let E be the event that right-hand box is emptied while the left-hand box still contains k matches. Then, using a negative binomial random variable with $p = \frac{1}{2}$, $r = N$, and $n = 2N - k$, we see that $P(E) = \binom{2N-k-1}{N-1} \left(\frac{1}{2}\right)^{2N-k}$. Now the desired probability is $2P(E)$.

Problem 78 Let E be the event that a single drawing results in two white and two black balls. Then $P(E) = \frac{\binom{4}{2} \binom{4}{2}}{\binom{8}{4}} = \frac{18}{35}$.

Let X be the number of selections until E occurs. Then

$$P(X = n) = \frac{17^{n-1} \cdot 18}{35^n}.$$

Problem 79 (a) $P(X = 0) = \frac{\binom{94}{10}}{\binom{100}{10}} = 0.5223$

(b)

$$\begin{aligned} P(X > 2) &= 1 - P(X = 0) - P(X = 1) - P(X = 2) \\ &= \frac{\binom{100}{10} - \binom{94}{10} - \binom{6}{1} \binom{94}{9} - \binom{6}{2} \binom{94}{8}}{\binom{100}{10}} = 0.0126 \end{aligned}$$

Problem 84 (a) For $i = 1, \dots, 5$, let $X_i = 1$ if the i -th box is empty and $X_i = 0$ otherwise. Then $X = X_1 + \dots + X_5$ is the number of empty boxes. For $i = 1, \dots, 5$,

$$E[X_i] = P(X_i = 1) = (1 - p_i)^{10}.$$

Thus

$$E[X] = E[X_1] + \cdots + E[X_5] = \sum_{i=1}^5 (1 - p_i)^{10}.$$

(b) For $i = 1, \dots, 5$, let $Y_i = 1$ if the i -th box has exactly 1 ball and $Y_i = 0$ otherwise. Then $Y = Y_1 + \cdots + Y_5$ is the number of boxes that have exactly 1 ball. For $i = 1, \dots, 5$,

$$E[Y_i] = P(Y_i = 1) = 10p_i(1 - p_i)^9.$$

Thus

$$E[Y] = E[Y_1] + \cdots + E[Y_5] = \sum_{i=1}^5 10p_i(1 - p_i)^9.$$

Problem 85 For $i = 1, \dots, k$, let $X_i = 1$ if the i -th type appear at least once in the set of n coupons. Then $X = X_1 + \cdots + X_k$ is the number of distinct types that appear in this set. For $i = 1, \dots, k$,

$$E[X_i] = P(X_i = 1) = 1 - P(X_i = 0) = 1 - (1 - p_i)^n.$$

Thus

$$E[X] = E[X_1] + \cdots + E[X_k] = k - \sum_{i=1}^k (1 - p_i)^n.$$

Chapter 5

Problem 1 (a) We have $1 = \int_{-1}^1 c(1 - x^2)dx = cx \left(1 - \frac{x^2}{3}\right) \Big|_{-1}^1 = \frac{4}{3}c$, so that $c = \frac{3}{4}$.

(b) We have $\int_{-1}^x f(y)dy = \frac{3}{4}y \left(1 - \frac{y^2}{3}\right) \Big|_{-1}^x = \frac{1}{2} + \frac{3}{4}x \left(1 - \frac{x^2}{3}\right)$ if $-1 \leq x \leq 1$. Hence,

$$F(x) = \begin{cases} 0 & x < -1, \\ \frac{1}{2} + \frac{3}{4}x \left(1 - \frac{x^2}{3}\right) & -1 \leq x \leq 1, \\ 1 & x > 1. \end{cases}$$

Problem 2 Determine C : $\int_0^\infty xe^{-\frac{x}{2}}dx = -2xe^{-\frac{x}{2}} \Big|_0^\infty + \int_0^\infty 2e^{-\frac{x}{2}}dx = (-2x - 4)e^{-\frac{x}{2}} \Big|_0^\infty = 4$, so that $C = \frac{1}{4}$.

Now, we have $P(X \geq 5) = \int_5^\infty \frac{1}{4}xe^{-\frac{x}{2}} = -\left(\frac{x}{2} + 1\right)e^{-\frac{x}{2}} \Big|_5^\infty = \frac{7}{2}e^{-\frac{5}{2}}$

Problem 4 (a) $P(X > 20) = \int_{20}^{\infty} \frac{10}{x^2} dx = -\frac{10}{x} \Big|_{20}^{\infty} = \frac{1}{2}$.

(b)

$$F(x) = \begin{cases} 0 & x < 10 \\ 1 - \frac{10}{x} & x \geq 10 \end{cases}$$

(c) Let's assume that lifetimes of the six devices are independent of each other. Let $p = 1 - F(15)$. Then the desired probability is

$$\sum_{i=3}^6 \binom{6}{i} p^i (1-p)^{6-i}.$$

Problem 5 We want to find C such that $F(C) \geq 0.99$. We have $F(C) = \int_0^C 5(1-x)^4 dx = -(1-x)^5 \Big|_0^C = 1 - (1-C)^5$. We want $1 - (1-C)^5 \geq 0.99$, i.e., $(1-C)^5 \leq 0.01$, hence $C \geq 1 - (0.01)^{0.2}$.