Third Homework Set — Solutions
Chapter 3

Problem 20 (a) $P(F|C) = \frac{P(FC)}{P(C)} = \frac{.02}{.05} = \frac{2}{5}$.

(b) $P(C|F) = \frac{P(FC)}{P(F)} = \frac{.02}{.52} = \frac{1}{26}$.

Problem 23 (a) Let W be the event that the ball selected urn II is white, E be the event that the transfered ball is white and F be the event that the transfered ball is red, then

$$P(W) = P(E)P(W|E) + P(F)P(W|F) = \frac{2}{3}\frac{1}{3} + \frac{1}{3}\frac{2}{3} = \frac{4}{9}.$$

(b)

$$P(E|W) = \frac{P(EW)}{P(W)} = \frac{P(E)P(W|E)}{P(W)} = \frac{1}{2}.$$

Problem 30 Let B and W be the events that the marble is black and white respectively, and let B_i be the event that box i is chosen. Then

$$P(B) = P(B_1)P(B|B_1) + P(B_2)P(B|B_2) = \frac{1}{2}\frac{1}{2} + \frac{1}{2}\frac{2}{3} = \frac{2}{12},$$

$$P(B_1|W) = \frac{P(B_1W)}{P(W)} = \frac{P(B_1)P(W|B_1)}{P(W)} = \frac{\frac{1}{2}\frac{1}{2}}{\frac{5}{12}} = \frac{3}{5}.$$

Problem 47 (a)

$$P(\text{all white}) = \frac{1}{6}\left(\frac{5}{15}+\frac{5}{15}\frac{4}{14}+\frac{5}{15}\frac{4}{14}\frac{3}{13}+\frac{5}{15}\frac{4}{14}\frac{3}{13}\frac{2}{12}+\frac{5}{15}\frac{4}{14}\frac{3}{13}\frac{2}{12}\frac{1}{11}\right).$$

(b)

$$P(3|\text{all white}) = \frac{\frac{1}{6}\frac{5}{15}\frac{4}{14}\frac{3}{13}}{P(\text{all white})}.$$

Problem 51 Let R be the event that she receives a job offer, S be the event that event of a strong recommendation, M the event of a moderate recommendation and W the event of a weak recommendation.

(a)

$$\begin{aligned} P(R) &= P(S)P(R|S) + P(M)P(R|M) + P(W)P(R|W) \\ &= (.8)(.7) + (.4)(.2) + (.1)(.1) = .65. \end{aligned}$$

(b)

$$P(S|R) = \frac{P(SR)}{P(R)} = \frac{P(S)P(R|S)}{P(R)} = \frac{(.8)(.7)}{.65} = \frac{56}{65}.$$

Similarly

$$P(M|R) = \frac{8}{65}, \quad P(W|R) = \frac{1}{65}.$$

Problem 56

$$P(\text{new}) = \sum_{i=1}^{m} p_i P(\text{new}|\text{the } n\text{-th coupon is of type i}) = \sum_{i=1}^{m} p_i (1-p_i)^{n-1}.$$

Problem 57 (a) $P\,(\text{original price after two days}) = \binom{2}{1} p(1-p) = 2p(1-p)$

(b) $P\,(\text{increase by one after three days}) = \binom{3}{2} p^2(1-p) = 3p^2(1-p)$

(c) $P\,(\text{increase on first day}|\text{increase by one after three days}) = \frac{p \cdot 2p(1-p)}{3p^2(1-p)} = \frac{2}{3}$

Problem 59 (a) $P\,(HHHH) = p^4$

(b) $P\,(THHH) = p^3(1-p)$

(c) The pattern $HHHH$ can *only* occur before $THHH$ if the first four coin flips come up heads. Hence, $P\,(THHH \text{ occurs before } HHHH) = 1 - p^4$.

Problem 64 Let E be the event that the wife answers correctly, and let F be the event that the husband answers correctly.

(a) If only one of them answers, then the probability of a correct answer is $P\,(E) = P\,(F) = p$.

(b) $P\,(\text{correct answer}) = P\,(EF) + \frac{1}{2} \cdot 2 \cdot p(1-p) = p^2 + p - p^2 = p$

Problem 66 Let E_i be the event that the i-th switch is on.

(a)

$$
\begin{aligned}
&P\left(\text{current flows from } A \text{ to } B\right)\\
&= \left(P\left(E_1E_2\right) + P\left(E_3E_4\right) - P\left(E_1E_2E_3E_4\right)\right) P\left(E_5\right)\\
&= (p_1p_2 + p_3p_4 - p_1p_2p_3p_4)p_5
\end{aligned}
$$

(b)

$$
\begin{aligned}
&P\left(\text{current flows from } A \text{ to } B\right)\\
&= P\left(E_1E_4 \cup E_1E_3E_5 \cup E_2E_5 \cup E_2E_3E_4\right)\\
&= p_1p_4 + p_1p_3p_5 + p_2p_5 + p_2p_3p_4\\
&-p_1p_3p_4p_5 - p_1p_2p_4p_5 - p_1p_2p_3p_4\\
&-p_1p_2p_3p_5 - p_1p_2p_3p_4p_5 - p_2p_3p_4p_5\\
&+4p_1p_2p_3p_4p_5 - p_1p_2p_3p_4p_5\\
&= p_1p_4 + p_1p_3p_5 + p_2p_5 + p_2p_3p_4\\
&-p_1p_3p_4p_5 - p_1p_2p_4p_5 - p_1p_2p_3p_4\\
&-p_1p_2p_3p_5 - p_2p_3p_4p_5 + 2p_1p_2p_3p_4p_5
\end{aligned}
$$

Problem 78 (a) $P\left(\text{exactly four games are played}\right) = P\left(ABAA\right) + P\left(BAAA\right) + P\left(ABBB\right) + P\left(BABB\right) = 2p^3(1-p) + 2p(1-p)^3 = 2p(1-p)\left(p^2 + (1-p)^2\right) = 2p(1-p)(1-2p+2p^2)$

(b) Let E be the event that A wins the match. Conditioning on the first two games of the match, we get $P\left(E\right) = P\left(E|A,A\right)p^2 + P\left(E|A,B\right)p(1-p) + P\left(E|B,A\right)(1-p)p + P\left(E|B,B\right)(1-p)^2 = p^2 + 2P\left(E\right)p(1-p)$ because $P\left(E|A,B\right) = P\left(E|B,A\right) = P\left(E\right)$. Hence, $P\left(E\right) = \frac{p^2}{1-2p(1-p)}$.

Problem 81 Using the gambler's ruin formula, the anwser is

$$\frac{1-(9/11)^{15}}{1-(9/11)^{30}}.$$

Problem 83 (a) Conditioning on the coin flip

$$P(\text{throw n is red}) = \frac{1}{2}\frac{4}{6} + \frac{1}{2}\frac{2}{6} = \frac{1}{2}.$$

(b)

$$P(R_3|R_1R_2) = \frac{P(R_1R_2R_3)}{P(R_1R_2)} = \frac{\frac{1}{2}(\frac{2}{3})^3 + \frac{1}{2}(\frac{1}{3})^3}{\frac{1}{2}(\frac{2}{3})^2 + \frac{1}{2}(\frac{1}{3})^2} = \frac{3}{5}.$$

(c)

$$P(A|R_1R_2) = \frac{P(R_1R_2|A)P(A)}{P(R_1R_2)} = \frac{(\frac{2}{3})^2\frac{1}{2}}{(\frac{2}{3})^2\frac{1}{2} + (\frac{1}{3})^2\frac{1}{2}} = \frac{4}{5}.$$

Problem 84 (a)

$$\begin{aligned}
P(A \text{ win}) &= \sum_{i=0}^{\infty} P(\text{ the first white appears on draw number } 3i+1) \\
&= \sum_{i=0}^{\infty} (\frac{2}{3})^{3i}\frac{1}{3} = \frac{1}{3}\sum_{i=0}^{\infty}(\frac{8}{27})^i = \frac{1}{3}\frac{1}{1-\frac{8}{27}} \\
P(B \text{ win}) &= \sum_{i=0}^{\infty} P(\text{ the first white appears on draw number } 3i+2) \\
&= \sum_{i=0}^{\infty} (\frac{2}{3})^{3i+1}\frac{1}{3} = \frac{2}{9}\sum_{i=0}^{\infty}(\frac{8}{27})^i = \frac{2}{9}\frac{1}{1-\frac{8}{27}} \\
P(C \text{ win}) &= \sum_{i=0}^{\infty} P(\text{ the first white appears on draw number } 3i+3) \\
&= \sum_{i=0}^{\infty} (\frac{2}{3})^{3i+2}\frac{1}{3} = \frac{4}{27}\sum_{i=0}^{\infty}(\frac{8}{27})^i = \frac{4}{27}\frac{1}{1-\frac{8}{27}}
\end{aligned}$$

(b)

$$\begin{aligned}
P(A \text{ win}) &= \frac{4}{12} + \frac{8}{12}\frac{7}{11}\frac{6}{10}\frac{4}{9} + \frac{8}{12}\frac{7}{11}\frac{6}{10}\frac{5}{9}\frac{4}{8}\frac{3}{7}\frac{4}{6} \\
P(B \text{ win}) &= \frac{8}{12}\frac{4}{11} + \frac{8}{12}\frac{7}{11}\frac{6}{10}\frac{5}{9}\frac{4}{8} + \frac{8}{12}\frac{7}{11}\frac{6}{10}\frac{5}{9}\frac{4}{8}\frac{3}{7}\frac{2}{6}\frac{4}{5} \\
P(C \text{ win}) &= \frac{8}{12}\frac{7}{11}\frac{4}{10} + \frac{8}{12}\frac{7}{11}\frac{6}{10}\frac{5}{9}\frac{4}{8}\frac{4}{7} + \frac{8}{12}\frac{7}{11}\frac{6}{10}\frac{5}{9}\frac{4}{8}\frac{3}{7}\frac{2}{6}\frac{1}{5}.
\end{aligned}$$

Chapter 4

Problem 1 Possible values of X: 0,2,4,-1,-2,1 Probabilities:

$$P(X=0) = \frac{\binom{2}{2}}{\binom{14}{2}} = \frac{1}{91}$$
$$P(X=2) = \frac{\binom{4}{1}\binom{2}{1}}{\binom{14}{2}} = \frac{8}{91}$$
$$P(X=4) = \frac{\binom{4}{2}}{\binom{14}{2}} = \frac{6}{91}$$
$$P(X=-1) = \frac{\binom{8}{1}\binom{2}{1}}{\binom{14}{2}} = \frac{16}{91}$$
$$P(X=-2) = \frac{\binom{8}{2}}{\binom{14}{2}} = \frac{28}{91}$$
$$P(X=1) = \frac{\binom{8}{1}\binom{4}{1}}{\binom{14}{2}} = \frac{32}{91}$$

Problem 4

$$P(X=1) = \frac{\binom{5}{1}9!}{10!} = \frac{1}{2}$$
$$P(X=2 = \frac{\binom{5}{1}\binom{5}{1}8!}{10!} = \frac{5}{18}$$
$$P(X=3) = \frac{\binom{5}{2}2!\binom{5}{1}7!}{10!} = \frac{5}{36}$$
$$P(X=4) = \frac{\binom{5}{3}3!\binom{5}{1}6!}{10!} = \frac{5}{84}$$
$$P(X=5) = \frac{\binom{5}{4}4!\binom{5}{1}5!}{10!} = \frac{5}{252}$$
$$P(X=6) = \frac{\binom{5}{5}5!\binom{5}{1}4!}{10!} = \frac{1}{252}$$
$$P(X=7) = P(X=8) = P(X=9) = P(X=10) = 0$$

Problem 5 The possible values are $n, n-2, n-4, ..., -n+4, -n+2, -n$.