Math 461 Test 2, Spring 2005

Calculators, books, notes and extra papers are *not* allowed on this test!

Show all work to qualify for full credits

1. (10 points) The time (in hours) required to repair a machine is an exponential random variable with parameter $\lambda = \frac{1}{2}$. Find (a) the probability that the repair time exceeds 2 hours; (b) the conditional probability that the repair time is at least 10 hours, given that its duration exceeds 9 hours.

2. (10 points) A certain basketball player knows that on average he will make 80 percent of his free throw attempts. Use normal approximation to find the probability that in 100 attempts he will be successful at least 90 times.

3. (10 points) Teams A and B play a series of games; the series will end when one of the teams wins 4 games. Suppose that team A wins each game with probability $\frac{2}{3}$, independent of the outcomes of the other games. Find the probability that a total of 6 games are played.

4. (15 points) Let X and Y be independent random variables each uniformly distributed on $(0, 1)$. Find density function of $Z = |X - Y|$.

5. (15 points) Suppose X and Y are independent geometric random variables with parameter $p = \frac{1}{3}$. Find $P(X \leq Y)$.

6. (10 points) Suppose that X is uniformly distributed over the interval $(-1, 1)$. Find the density of $Y = -\ln(1 - |X|)$.

7. (20 points) The joint density of X and Y is given by

$$f(x, y) = \begin{cases} x + y, & 0 < x < 1, 0 < y < 1 \\ 0, & \text{otherwise .} \end{cases}$$

(a) Find the marginal density of X. (b) Find EX and $\text{Var}(X)$.

8. (10 points) The gross daily sales at a certain convenient store is a normal random variable with mean \$2000 and standard deviation \$200. Assume that sales are independent from day to day. Find the probability that the total gross sales in the next 4 days exceeds \$8800.

1. (a) $P(X > 2) = \int_2^\infty \frac{1}{2}e^{-\frac{x}{2}}dx = e^{-1}$..

(b) Using the memoryless property of exponential random variables, we get

$$P(X > 10|X > 9) = P(X > 1) = \int_1^\infty \frac{1}{2}e^{-\frac{x}{2}}dx = e^{-\frac{1}{2}}.$$

2. Let X be the number of times that he will be successful in 100 attempts. Then X is a binomial random variable with parameter $n = 100$ and $p = .8$. Thus, by using normal approximation, we get

$$P(X \geq 90) = P(X \geq 89.5) = P(\frac{X - 80}{4} \geq \frac{9.5}{4}) \approx 1 - \Phi(2.37) = .0089.$$

3. Let A be the event that team A wins in 6 games, B be the event that B wins in 6 games, and C be the event that a total of 6 games are played. Then

$$P(C) = P(A) + P(B) = \binom{5}{3}(\frac{2}{3})^4(\frac{1}{3})^2 + \binom{5}{3}(\frac{1}{3})^4(\frac{2}{3})^2.$$

4. Z is a random variable taking values in $(0, 1)$. For any $z \in (0, 1)$, by geometrical considerations, we can easily get

$$P(Z \leq z) = P(|X - Y| \leq z) = 1 - (1 - z)^2.$$

Thus the density of Z is

$$f_Z(z) == \begin{cases} 2(1-z), & z \in (0,1) \\ 0, & \text{otherwise.} \end{cases}$$

5.

$$P(X \leq Y) = \sum_1^\infty P(X = k, X \leq Y) = \sum_{k=1}^\infty P(X = k, Y \geq k) = \sum_{k=1}^\infty P(X = k)P(Y \geq k)$$
$$= \sum_{k=1}^\infty (\frac{2}{3})^{k-1}\frac{1}{3}(\frac{2}{3})^{k-1} = \frac{1}{3}\sum_{k=1}^\infty (\frac{4}{9})^{k-1} = \frac{1}{3}\sum_{j=0}^\infty (\frac{4}{9})^j = \frac{1}{3} \cdot \frac{9}{5} = \frac{3}{5}.$$

6. Y is a positive random variable. For any $y > 0$,

$$P(Y \leq y) = P(-\ln(1 - |X|) \leq y) = P(1 - |X| \geq e^{-y}) = P(|X| \leq 1 - e^{-y}) = 1 - e^{-y}.$$

Thus the density of Y is given

$$f_Y(y) = \begin{cases} e^{-y}, & y > 0 \\ 0, & y \leq 0. \end{cases}$$

7. (a). For any $x \in (0, 1)$,

$$f_X(x) = \int_0^1 (x + y)dy = x + \frac{1}{2}.$$

Thus the marginal density of X is

$$f_X(x) = \begin{cases} x + \frac{1}{2}, & x \in (0,1) \\ 0, & \text{otherwise.} \end{cases}$$

(b) $EX = \int_0^1 x(x + \frac{1}{2})dx = \frac{7}{12}$, $E(X^2) = \int_0^1 x^2(x + \frac{1}{2})dx = \frac{5}{12}$. Thus

$$\text{Var}(X) = E(X^2) - (EX)^2 = \frac{11}{144}.$$

8. From the assumption of the problem we know that the total gross sales X in the next 4 days is a normal random variable with $\mu = 8000$ and $\sigma^2 = 4 \cdot (200)^2$. Thus

$$P(X \geq 8800) = P(\frac{X - 8000}{400} \geq 2) = 1 - \Phi(2) = .0228.$$