Math 461 Test 1, Spring 2005

Calculators, books, notes and extra papers are *not* allowed on this test!

Show all work to qualify for full credits

1. (20 points) An urn contains 4 red, 6 blue and 10 green balls. 3 balls are randomly selected from the urn, find the probability that they are all of the same color if (a) the balls are drawn without replacement; (b) the balls are drawn with replacement.

Solution (a) If the balls are drawn without replacement, then

$$\begin{aligned} P(\text{ all 3 are of the same color}) &= \frac{\binom{4}{3}+\binom{6}{3}+\binom{10}{3}}{\binom{20}{3}} \\ &= \frac{4\cdot 3\cdot 2+6\cdot 5\cdot 4+10\cdot 9\cdot 8}{20\cdot 19\cdot 18}. \end{aligned}$$

(b) If the balls are drawn with replacement, then

$$P(\text{ all 3 are of the same color}) = \frac{4^3+6^3+10^3}{20^3}.$$

2. (20 points) An 8-card hand is drawn without replacement from an ordinary deck of 52 cards. Find the probability that it contains the ace and king of at least one suit.

Solution Let A_1 be the event that the hand contains the ace and king of hearts, A_2 the event that the hand contains the ace and king of spades, A_3 the event that the hand contains the ace and king of clubs and A_4 the event that the hand contains the ace and king of diamonds. Then, by using the inclusion-exclusion formula, the probability that the hand contains the ace and king of at least one suit is equal to

$$P(\cup_{i=1}^4 A_i) = 4\frac{\binom{50}{6}}{\binom{52}{8}} - \binom{4}{2}\frac{\binom{48}{4}}{\binom{52}{8}} + \binom{4}{3}\frac{\binom{46}{2}}{\binom{52}{8}} - \frac{1}{\binom{52}{8}}.$$

3. (20 points) A box contains 7 red and 13 blue balls. Two balls are randomly selected (without replacement) and are discarded without their colors being seen. A third ball is drawn randomly. (a) Find the probability that the third ball is red. (b) Given that the third ball is red, find the probability that both discarded balls were blue.

Solution (a)

$$\begin{aligned} P(R_3) &= P(R_1R_2R_3) + P(R_1B_2R_3) + P(B_1R_2R_3) + P(B_1B_2R_3) \\ &= \frac{7}{20}\frac{6}{19}\frac{5}{18} + \frac{7}{20}\frac{13}{19}\frac{6}{18} + \frac{13}{20}\frac{7}{19}\frac{6}{18} + \frac{13}{20}\frac{12}{19}\frac{7}{18} = \frac{2}{20}. \end{aligned}$$

(b)

$$P(B_1B_2|R_3) = \frac{P(B_1B_2R_3)}{P(R_3)} = \frac{\frac{13}{20}\frac{12}{19}\frac{7}{18}}{\frac{2}{20}}.$$

4. (15 points) A parallel system functions whenever at least one of its components works. Consider a parallel system with 3 components and suppose that each component independently works with probability $\frac{2}{3}$. Given that the system is functioning, find the probability the first two components are both functioning.

Solution For $i = 1, 2, 3$, let A_i be the event that the i-th component is functioning and let A be the event that the system is functioning. Then $A = A_1 \cup A_2 \cup A_3$. Thus

$$P(A) = P(A_1 \cup A_2 \cup A_3) = 1 - P(A_1^c A_2^c A_3^c) = 1 - (\frac{1}{3})^3.$$

$$P(A_1A_2|A) = \frac{P(A_1A_2A)}{P(A)} = \frac{P(A_1A_2)}{P(A)} = \frac{(\frac{2}{3})^2}{1 - (\frac{1}{3})^3}.$$

5. (10 points) Let X be a random variable whose distribution function F is given by

$$F(x) = \begin{cases} 0, & x < 0, \\ x/4, & 0 \le x < 1, \\ 1/2 & 1 \le x < 2, \\ \frac{x}{12} + \frac{1}{2}, & 2 \le x < 3 \\ 1, & 3 \le x. \end{cases}$$

Find (a) $P(X < 2)$; (b) $P(X = 2)$; (c) $P(1 \le X < 3)$; (d) $P(X > 3/2)$; (e) $P(2 < X \le 7)$.

Solution (a) $P(X < 2) = F(2-) = \frac{1}{2}$.
(b) $P(X = 2) = F(2) - F(2-) = (\frac{2}{12} + \frac{1}{2}) - \frac{1}{2} = \frac{1}{6}$.
(c) $P(1 \le X < 3) = F(3-) - F(1-) = (\frac{3}{12} + \frac{1}{2}) - \frac{1}{4} = \frac{1}{2}$.
(d) $P(X > 3/2) = 1 - F(\frac{3}{2}) = 1 - \frac{1}{2} = \frac{1}{2}$.
(e) $P(2 < X \le 7) = F(7) - F(2) = 1 - (\frac{2}{12} + \frac{1}{2}) = \frac{1}{3}$.

6. (15 points) A card is drawn at random from an an ordinary deck of 52 cards and its face value is noted, and then this card is returned to the deck. This procedure is done 4 times all together. Let X be the total number of aces selected and $Y = \cos(\frac{\pi}{2}X)$
(a) Find $P(X \ge 1)$.
(b) Find the expectation and variance of Y.

Solution (a) $P(X \ge 1) = 1 - P(X = 0) = 1 - (\frac{12}{13})^4$.
(b)

$$EY = E[\cos(\frac{\pi}{2}X)] = \sum_{k=0}^{4} \cos(\frac{k\pi}{2}) \binom{4}{k} (\frac{2}{3})^k (\frac{1}{3})^{4-k} = (\frac{12}{13})^4 - \binom{4}{2} (\frac{1}{13})^2 (\frac{12}{13})^2 + (\frac{1}{13})^4.$$

$$E[Y^2] = E[\cos^2(\frac{\pi}{2}X)] = \sum_{k=0}^{4} \cos^2(\frac{k\pi}{2}) \binom{4}{k} (\frac{2}{3})^k (\frac{1}{3})^{4-k} = (\frac{12}{13})^4 + \binom{4}{2} (\frac{1}{13})^2 (\frac{12}{13})^2 + (\frac{1}{13})^4.$$

$$\text{Var}(Y) = E[Y^2] - (EY)^2 = (\frac{12}{13})^4 + \binom{4}{2} (\frac{1}{13})^2 (\frac{12}{13})^2 + (\frac{1}{13})^4 - \left((\frac{12}{13})^4 - \binom{4}{2} (\frac{1}{13})^2 (\frac{12}{13})^2 + (\frac{1}{13})^4 \right)^2.$$