Math 461 Test 1, Spring 2005

Calculators, books, notes and extra papers are *not* allowed on this test!

Show all work to qualify for full credits

1. (20 points) An urn contains 5 red, 6 blue and 8 green balls. 3 balls are randomly selected from the urn, find the probability that they are all of the same color if (a) the balls are drawn without replacement; (b) the balls are drawn with replacement.

Solution (a) If the balls are drawn without replacement, then

$$\begin{aligned} P(\text{ all 3 are of the same color}) &= \frac{\binom{5}{3}+\binom{6}{3}+\binom{8}{3}}{\binom{19}{3}} \\ &= \frac{5\cdot 4\cdot 3+6\cdot 5\cdot 4+8\cdot 7\cdot 6}{19\cdot 18\cdot 17}. \end{aligned}$$

(b) If the balls are drawn with replacement, then

$$P(\text{ all 3 are of the same color}) = \frac{5^3+6^3+8^3}{19^3}.$$

2. (15 points) An 7-card hand is drawn without replacement from an ordinary deck of 52 cards. Find the probability that it contains the ace and king of at least one suit.

Solution Let A_1 be the event that the hand contains the ace and king of hearts, A_2 the event that the hand contains the ace and king of spades, A_3 the event that the hand contains the ace and king of clubs and A_4 the event that the hand contains the ace and king of diamonds. Then, by using the inclusion-exclusion formula, the probability that the hand contains the ace and king of at least one suit is equal to

$$P(\cup_{i=1}^4 A_i) = 4\frac{\binom{50}{5}}{\binom{52}{7}} - \binom{4}{2}\frac{\binom{48}{3}}{\binom{52}{7}} + \binom{4}{3}\frac{\binom{46}{1}}{\binom{52}{7}}.$$

3. (20 points) A box contains 7 red and 13 blue balls. Two balls are randomly selected (without replacement) and are discarded without their colors being seen. A third ball is drawn randomly. (a) Find the probability that the third ball is red. (b) Given that the third ball is red, find the probability that both discarded balls were blue.

Solution (a)

$$\begin{aligned} P(R_3) &= P(R_1R_2R_3) + P(R_1B_2R_3) + P(B_1R_2R_3) + P(B_1B_2R_3) \\ &= \frac{7}{20}\frac{6}{19}\frac{5}{18} + \frac{7}{20}\frac{13}{19}\frac{6}{18} + \frac{13}{20}\frac{7}{19}\frac{6}{18} + \frac{13}{20}\frac{12}{19}\frac{7}{18} = \frac{7}{20}. \end{aligned}$$

(b)

$$P(B_1B_2|R_3) = \frac{P(B_1B_2R_3)}{P(R_3)} = \frac{\frac{13}{20}\frac{12}{19}\frac{7}{18}}{\frac{7}{20}}.$$

4. (20 points) A circuit is given below. The probability of the i-th switch is on is $\frac{1}{i+1}$, $i = 1, 2, 3, 4$. Suppose that all switches function independently. (a) Find the probability that a current can flow from A to B. (b) Given that a current can flow from A to B, find the probability that both switches 1 and 2 are on.

Solution (a) Let A be the vent that a current can flow from A to B, then the probability of A is equal to

$$P((A_1 \cup A_2 \cup A_3)A_4) = P(A_1 \cup A_2 \cup A_3)P(A_4) = (1 - P(A_1^cA_2^cA_3^c))P(A_4) = (1 - \frac{1}{2}\frac{2}{3}\frac{3}{4})\frac{1}{5} = \frac{3}{20}.$$

(b)

$$P(A_1A_2|A) = \frac{P(A_1A_2A)}{P(A)} = \frac{P(A_1A_2A_4)}{P(A)} = \frac{\frac{1}{2}\frac{1}{3}\frac{1}{5}}{\frac{3}{20}} = \frac{2}{9}.$$

5. (10 points) Let X be a random variable whose distribution function F is given by

$$F(x) = \begin{cases} 0, & x < 0, \\ x/4, & 0 \le x < 1, \\ 1/2 & 1 \le x < 2, \\ \frac{x}{12} + \frac{1}{2}, & 2 \le x < 3 \\ 1, & 3 \le x. \end{cases}$$

Find (a) $P(X < 2)$; (b) $P(X = 2)$; (c) $P(1 \le X < 3)$; (d) $P(X > 3/2)$; (e) $P(2 < X \le 7)$.

Solution (a) $P(X < 2) = F(2-) = \frac{1}{2}$.
(b) $P(X = 2) = F(2) - F(2-) = (\frac{2}{12} + \frac{1}{2}) - \frac{1}{2} = \frac{1}{6}$.
(c) $P(1 \le X < 3) = F(3-) - F(1-) = (\frac{3}{12} + \frac{1}{2}) - \frac{1}{4} = \frac{1}{2}$.
(d) $P(X > 3/2) = 1 - F(\frac{3}{2}) = 1 - \frac{1}{2} = \frac{1}{2}$.
(e) $P(2 < X \le 7) = F(7) - F(2) = 1 - (\frac{2}{12} + \frac{1}{2}) = \frac{1}{3}$.

6. Independent trials, each results in a success with probability $\frac{2}{3}$, are performed 4 times. Let X be the total number of successes and $Y = \sin(\frac{\pi}{2}X)$
(a) (5 points) Find $P(X \ge 1)$.
(b) (10 points) Find the expectation and variance of Y.

Solution (a) $P(X \ge 1) = 1 - P(X = 0) = 1 - (\frac{1}{3})^4$.
(b)

$$EY = E[\sin(\frac{\pi}{2}X)] = \sum_{k=0}^{4} \sin(\frac{k\pi}{2}) \binom{4}{k} (\frac{2}{3})^k (\frac{1}{3})^{4-k} = \binom{4}{1} (\frac{2}{3})(\frac{1}{3})^3 - \binom{4}{1} (\frac{2}{3})^3(\frac{1}{3}) = -\frac{8}{27}.$$

$$E[Y^2] = E[\sin^2(\frac{\pi}{2}X)] = \sum_{k=0}^{4} \sin^2(\frac{k\pi}{2}) \binom{4}{k} (\frac{2}{3})^k (\frac{1}{3})^{4-k} = \binom{4}{1} (\frac{2}{3})(\frac{1}{3})^3 + \binom{4}{1} (\frac{2}{3})^3(\frac{1}{3}) = \frac{40}{81}.$$

$$\mathrm{Var}(Y) = E[Y^2] - (EY)^2 = \frac{40}{81} - (\frac{8}{27})^2.$$