Solutions to Math 461 Test 1, Fall 2021

1. (16 points) (a) Suppose that E_1, E_2, E_3, E_4, E_5 are independent events and that $\mathbb{P}(E_1) = \mathbb{P}(E_2) = \frac{1}{2}$, $\mathbb{P}(E_3) = \mathbb{P}(E_4) = \frac{1}{4}, \mathbb{P}(E_5) = \frac{1}{3}$. Find $\mathbb{P}(((E_1 \cap E_2) \cup (E_3 \cap E_4)) \cap E_5)$.
(b) Suppose that X is a geometric random variable with parameter $p = \frac{1}{4}$. Find $E[(2-X)^2]$.

Solution. (a)

$$\begin{aligned}
&\mathbb{P}(((E_1 \cap E_2) \cup (E_3 \cap E_4)) \cap E_5) = P((E_1 \cap E_2) \cup (E_3 \cap E_4))\mathbb{P}(E_5) \\
&= (\mathbb{P}(E_1 \cap E_2) + \mathbb{P}(E_3 \cap E_4) - \mathbb{P}(E_1 \cap E_2 \cap E_3 \cap E_4))\, \mathbb{P}(E_5) \\
&= (\mathbb{P}(E_1)\mathbb{P}(E_2) + \mathbb{P}(E_3)\mathbb{P}(E_4) - \mathbb{P}(E_1)\mathbb{P}(E_2)\mathbb{P}(E_3)\mathbb{P}(E_4))\, \mathbb{P}(E_5) \\
&= \left(\frac{1}{2}\frac{1}{2} + \frac{1}{4}\frac{1}{4} - \frac{1}{2}\frac{1}{2}\frac{1}{4}\frac{1}{4}\right)\frac{1}{3}.
\end{aligned}$$

(b)

$$\begin{aligned}
&E[(2-X)^2] = E[4 - 4X + X^2] \\
&= 4 - 4E[X] + E[X^2] = 4 - 4E[X] + (\mathrm{Var}(X) + (E[X])^2) \\
&= 4 - 4 \cdot 4 + \left(\frac{3/4}{(1/4)^2} + 4^2\right).
\end{aligned}$$

2. (16 points) A box contains 3 red and 5 black balls. Players A and B withdraw balls randomly from the box consecutively (and without replacement) until a red ball is selected; whoever get a red ball is declared the winner. Find the probability that A is the winner.

Solution.

$$\begin{aligned}
P(A \text{ wins after 1 selction}) &= \frac{3}{8} \\
P(A \text{ wins after 3 selctions}) &= \frac{5}{8} \cdot \frac{4}{7} \cdot \frac{3}{6} \\
P(A \text{ wins after 5 selctions}) &= \frac{5}{8} \cdot \frac{4}{7} \cdot \frac{3}{6} \cdot \frac{2}{5} \cdot \frac{3}{4} \\
P(A \text{ is the winner}) &= \frac{3}{8} + \frac{5}{8} \cdot \frac{4}{7} \cdot \frac{3}{6} + \frac{5}{8} \cdot \frac{4}{7} \cdot \frac{3}{6} \cdot \frac{2}{5} \cdot \frac{3}{4}.
\end{aligned}$$

3. (16 points) Box A contains 2 white and 4 red balls, whereas box B contains 3 white and 3 red balls. A ball is randomly chosen from box A and put into box B, and a ball is then randomly selected from box B. (a) Find the probability that the ball selected from box B is white; (b) find the conditional probability that the transferred ball was white given that a white ball is selected from box B.

Solution. Let E be the event that the transferred ball is white and let F be the event that the ball selected box B is white.

(a)

$$P(F) = P(E \cap F) + P(E^c \cap F) = P(E)P(F|E) + P(E^c)P(F|E^c)$$
$$= \frac{1}{3}\frac{4}{7} + \frac{2}{3}\frac{3}{7} = \frac{10}{21}.$$

(b)

$$P(E|F) = \frac{P(E \cap F}{P(F)} = \frac{P(E)P(F|E)}{P(F)} = \frac{\frac{1}{3}\frac{4}{7}}{\frac{10}{21}} = \frac{2}{5}.$$

4. (18 points) A box contains 10 orange, 10 blue, 10 green and 10 red balls. 8 balls are randomly selected from the box without replacement. Find the probability that at least one color is missing from the 8 selected balls.

Solution. Let A_1 be the event that orange is missing from the 8 selected balls; A_2 the event that blue is missing from the 8 selected balls; A_3 the event that green is missing from the 8 selected balls; A_4 the event that red is missing from the 8 selected balls. Then

$$P(A_1 \cup A_2 \cup A_3 \cup A_4)$$
$$= \sum_{i=1}^{4} P(A_i) - \sum_{1 \le i < j \le 4} P(A_i \cap A_j) + \sum_{1 \le i < j < k \le 4} P(A_i \cap A_j \cap A_k)$$
$$= 4 \cdot \frac{\binom{30}{8}}{\binom{40}{8}} - \binom{4}{2} \cdot \frac{\binom{20}{8}}{\binom{40}{8}} + \binom{4}{3} \cdot \frac{\binom{10}{8}}{\binom{40}{8}}.$$

5. (16 points) Two teams play a series of games. The series is finished as soon as one of the teams wins 4 games. Suppose that the two teams are evenly matched and each has probability 1/2 of winning each game. Let X be the number of games played in the series. (a) Find the mass function of X; (b) find $E[X]$.

Solution. (a)

$$P(X = 4) = 2(\frac{1}{2})^4 = \frac{1}{8},$$
$$P(X = 5) = 2\binom{4}{3}(\frac{1}{2})^5 = \frac{1}{4},$$
$$P(X = 6) = 2\binom{5}{3}(\frac{1}{2})^6 = \frac{5}{16},$$
$$P(X = 7) = 2\binom{6}{3}(\frac{1}{2})^7 = \frac{5}{16}.$$

(b) $E[X] = 4 \cdot \frac{1}{8} + 5 \cdot \frac{1}{4} + 6 \cdot \frac{5}{16} + 7 \cdot \frac{5}{16}$.

6. (18 points) There are 100 type of coupons. Each time you collect a coupon, it is equally likely to be any of the 100 types. You have collected 30 coupons. Find the expected number of distinct types in your collection.

Solution. For $i = 1, \ldots, 100$, let $X_i = 1$ if there is at least one type i coupon in your collection and $X_i = 0$ otherwise. Then $X_1 + \cdots + X_{100}$ is the number of distinct types in your collection. For $i = 1, \ldots, 100$,

$$P(X_i = 1) = 1 - \left(\frac{99}{100}\right)^{30}.$$

Thus

$$E[X_1 + \cdots + X_{100}] = E[X_1] + \cdots + E[X_{100}] = 100 \cdot \left(1 - \left(\frac{99}{100}\right)^{30}\right).$$