Math 461 Fall 2021

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Outline

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Solution to Test 2 is is on my homepage. The distribution of scores for Test 2 is also available on my homepage.

HW10 is due Friday, 11/19, before the end of class. You can either turn in a hard copy or submit your HW10 electronically as a pdf file via the Moodle page. Make sure that your HW10 is uploaded successfully.

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Central limit theorem

Suppose that $X_1, X_2, \ldots$ are independent and identically distributed random variables with common mean μ and common variance σ^2. Then the distribution of

$$\frac{X_1 + \cdots + X_n - n\mu}{\sigma\sqrt{n}}$$

tends to the standard normal distribution as $n \to \infty$. That is, for any $a \in \mathbb{R}$,

$$P\left(\frac{X_1 + \cdots + X_n - n\mu}{\sigma\sqrt{n}} \leq a\right) \to \Phi(a), \quad \text{as } n \to \infty.$$

Note that the generality of the theorem above. The common distributions of $X_1, X_2, \ldots$ can be discrete, can be continuous, and can be neither discrete nor continuous. It can be regarded as universality law.

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The key to the proof of the central limit theorem is the following result, which we state without proof.

Proposition

Let $Z_1, Z_2, \ldots$ be a sequence of random variables with distribution functions F_{Z_n} and moment generating functions M_{Z_n}, $n \geq 1$; let Z be a random variable with distribution function F_Z and moment generating function M_Z. If $M_{Z_n}(t) \to M_Z(t)$ for all t, then $F_{Z_n}(z) \to F_Z(z)$ for all z at which F_Z is continuous.

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In general, we can not strengthen the conclusion of the proposition to $F_{Z_n}(z) \to F_Z(z)$ for all z. Here is an example: Take $Z_n = \frac{1}{n}$ and $Z = 0$. Then as $n \to \infty$,

$$M_{Z_n}(t) = e^{t/n} \to 1 = M_Z(t), \quad t \in \mathbb{R}.$$

But

$$\lim_{n\to\infty} F_{Z_n}(0) = 0 \neq 1 = F_Z(0).$$

0 is a discontinuous point of F_Z.

When the limit distribution function F_Z is continuous (this is the case when the limit distribution is a normal distribution), we do have $F_{Z_n}(z) \to F_Z(z)$ for all z since there are no discontinuous points.

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Proof of the central limit theorem

By considering $X_n' = (X_n - \mu)/\sigma$ if necessary, we may assume without loss of generality that $\mu = 0$ and $\sigma^2 = 1$. We will prove the central limit theorem under the extra assumption that the moment generating function $M(t)$ of X_1 exists and is finite for all $t \in \mathbb{R}$.

The moment generating function $X_i/\sqrt{n}$ is

$$E\left[e^{tX_i/\sqrt{n}}\right] = M\left(\frac{t}{\sqrt{n}}\right), \quad t \in \mathbb{R},$$

and thus the moment generating function of $\sum_{i=1}^n X_i/\sqrt{n}$ is

$$\left[M\left(\frac{t}{\sqrt{n}}\right)\right]^n.$$

Proof of the central limit theorem

Since $\mu = 0$ and $\sigma^2 = 1$, we have $M'(0) = 0$ and $M''(0) = 1$, thus by the Taylor formula, we have

$$M(t) = 1 + \frac{1}{2}t^2 + o(t^2).$$

Thus

$$M\left(\frac{t}{\sqrt{n}}\right) = 1 + \frac{t^2}{2n} + o(\frac{t^2}{n}), \quad t \in \mathbb{R}.$$

Consequently, for any $t \in \mathbb{R}$,

$$\left[M\left(\frac{t}{\sqrt{n}}\right)\right]^n = \left(1 + \frac{t^2}{2n} + o(\frac{t^2}{n})\right)^n \to e^{t^2/2}$$

which is the moment generating function of the standard normal distribution. The proof is now complete.

Example 1

The number of students who enroll in a psychology course is a Poisson random variable with parameter 100. The professor in charge of the course has decided that if the number enrolling is 120 or more he will teach the course in two separate sections, whereas if fewer than 120 students enroll he will teach all students together in a single section. Use the central limit theorem to approximate the probability that the professor will have to teach two sections.

The exact answer is

$$e^{-100} \sum_{i=120}^{\infty} \frac{100^i}{i!}$$

which is not easy to evaluate.

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There is only one random variable involved here. How can we use the central limit theorem?

Recall that a Poisson random variable with parameter 100 can written as the sum of 100 independent Poisson random variables with parameter 1. With this, we can use the central limit theorem.

$$P(X \geq 120) = P(X \geq 119.5) = P\left(\frac{X - 100}{\sqrt{100}} \geq \frac{119.5 - 100}{\sqrt{100}}\right)$$
$$\approx 1 - \Phi(1.95) \approx .0256.$$

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Example 2

If 10 fair dice are rolled, find the approximate probability that sum obtained is between 20 and 40 inclusive.

Let X_i denote the value of the i-th die, $i = 1, \ldots, 10$. $E[X_i] = \frac{7}{2}$ and $\mathrm{Var}(X_i) = \frac{35}{12}$. The central limit theorem yields

$$P(30 \le X \le 40) = P(29.5 \le X \le 40.5)$$
$$= P\left(\frac{29.5 - 35}{\sqrt{350/12}} \le \frac{X - 35}{\sqrt{350/12}} \le \frac{40.5 - 35}{\sqrt{350/12}}\right)$$
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Example 3

Let $X_i, i = 1, \ldots, 10$ be independent random variables, each uniformly distributed in the interval $(0, 1)$. Approximate the probability $P(\sum_{i=1}^{10} X_i > 6)$.

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$$P\left(\sum_{i=1}^{10} X_i > 6\right) = P\left(\frac{\sum_{i=1}^{10} X_i - 5}{\sqrt{10/12}} > \frac{6-5}{\sqrt{10/12}}\right)$$

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