Math 461 Fall 2021

Renming Song

University of Illinois at Urbana-Champaign

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Outline

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HW9 is due Friday, 11/05, before the end of class. Yo can submit your HW9 via the course Moodle page. Make make that your HW is uploaded successfully

Solution to HW8 is on my homepage.

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Recall that if X and Y are discrete random variables with joint mass function $p(\cdot,\cdot)$, and if y is such that $p_Y(y) = P(Y = y) > 0$, then the conditional mass function of X given $Y = y$ is given by

$$p_{X|Y}(x|y) = \frac{p(x,y)}{p_Y(y)}, \quad x \in \mathbb{R}.$$

Thus it is natural to define to the conditional expectation of X given $Y = y$ to be

$$E[X|Y = y] = \sum_x x p_{X|Y}(x|y)$$

and the conditional expectation of $\phi(X)$ given $Y = y$ to be

$$E[\phi(X)|Y = y] = \sum_x \phi(x) p_{X|Y}(x|y).$$

For any y with $p_Y(y) = P(Y = y) = 0$, we define

$$E[X|Y = y] = 0, \quad E[\phi(X)|Y = y] = 0.$$

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Example 1

Suppose that X and Y are independent Poisson random variables with parameters λ_1 and λ_2 respectively. For $n = 1, 2, \ldots$, find (a) $E[X|X+Y=n]$; (b) $E[X^2|X+Y=n]$.

We know that, given $X+Y=n$, X is a binomial random variable with parameters $(n, \lambda_1/(\lambda_1+\lambda_2))$. Thus

$$E[X|X+Y=n] = \frac{n\lambda_1}{\lambda_1+\lambda_2}$$

and

$$E[X^2|X+Y=n] = \frac{n\lambda_1\lambda_2}{(\lambda_1+\lambda_2)^2} + \frac{n^2\lambda_1^2}{(\lambda_1+\lambda_2)^2}.$$

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Example 2

Suppose that X and Y are independent geometric random variables with parameter p. For $n = 2, 3, \ldots,$ find (a) $E[X|X+Y=n]$; (b) $E[X^2|X+Y=n]$.

We know that, given $X+Y=n$, X takes the values $1, \ldots, n-1$ with probability $\frac{1}{n-1}$ each. Thus

$$E[X|X+Y=n] = \frac{n}{2}$$

and

$$E[X^2|X+Y=n] = \sum_{i=1}^{n-1} i^2 \frac{1}{n-1} = \frac{n(2n-1)}{6}.$$

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Recall that if X and Y are jointly absolutely continuous with joint density function $f(\cdot,\cdot)$, and if y is such that $f_Y(y) > 0$, then the conditional density function of X given $Y = y$ is given by

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Example 3

The joint density of X and Y is given by

$$f(x,y) = \begin{cases} \frac{1}{y} e^{-x/y} e^{-y}, & x > 0, y > 0; \\ 0, & \text{otherwise} \end{cases}$$

For $y > 0$, find (a) $E[X|Y = y]$; (b) $E[X^2|Y = y]$.

For $y > 0$,

$$f_Y(y) = \int_0^\infty \frac{1}{y} e^{-x/y} e^{-y} dx = e^{-y}.$$

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Hence

$$E[X|Y = y] = y$$

and

$$E[X^2|Y = y] = y^2 + y^2 = 2y^2.$$

Example 4

Suppose that U_1, U_2 are independent uniform random variables on $(0, 1)$. Let $X = \min(U_1, U_2)$ and $Y = \max(U_1, U_2)$. For $y \in (0, 1)$, find $E[X|Y = y]$.

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The joint density of X and Y is

$$f(x,y) = \begin{cases} 2, & 0 < x < y < 1, \\ 0, & \text{otherwise.} \end{cases}$$

The density of Y is

$$f_Y(y) = \begin{cases} 2y, & 0 < y < 1, \\ 0, & \text{otherwise.} \end{cases}$$

Thus, for $y \in (0,1)$, the conditional density of X given $Y = y$ is

$$f_{X|Y}(x|y) = \begin{cases} 1/y, & 0 < x < y, \\ 0, & \text{otherwise.} \end{cases}$$

That is, given $Y = y \in (0,1)$, X is uniform on $(0,y)$. Hence

$$E[X|Y=y] = \frac{y}{2}.$$

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$$E[X|Y = y] = \frac{y}{2}.$$

Now we define

$$E[X|Y] = \phi(Y)$$

where

$$\phi(y) = E[X|Y = y], \quad y \in \mathbb{R}.$$

Thus $E[X|Y]$ is a random variable.

Theorem

$$E[E[X|Y]] = E[X].$$

Let's give a proof in the case when X and Y are jointly absolutely continuous.

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Let's give a proof in the case when X and Y are jointly absolutely continuous.

Let

$$\phi(y) = E[X|Y = y] = \int_{-\infty}^{\infty} x f_{X|Y}(x|y)dx.$$

Then

$$\begin{aligned} E[E[X|Y]] =& E[\phi(Y)] = \int_{-\infty}^{\infty} \phi(y) f_Y(y)dy \\ =& \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} x f_{X|Y}(x|y)d \right) f_Y(y)dy \\ =& \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x f(x,y)dxdy \\ =& E[X]. \end{aligned}$$

Example 5

A miner is trapped in a mine containing 3 doors. The first door leads to a tunnel that will take him to safety after 3 hours of travel. The second door leads to a tunnel that will return him to the mine after 5 hours of travel. The third leads to a tunnel that will return him to the mine after 7 hours of travel. If we assume that the miner is at all times equally likely to choose any one of the three doors, what is the expected length of time until he reaches safety?

Let X be time, in hours, it takes him to reach safety. Finding the mass function of X and then using the definition to find $E[X]$ is not really an option. We need another method. Let Y denote the initial door he chooses. Then $E[E[X|Y]] = E[X]$.

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So

$$\begin{aligned}
E[X] &= E[E[X|Y]] \\
&= E[X|Y=1]P(Y=1) + E[X|Y=2]P(Y=2) + E[X|Y=3]P(Y=3) \\
&= \frac{1}{3}\left(E[X|Y=1] + E[X|Y=2] + E[X|Y=3]\right) \\
&= \frac{1}{3}\left(3 + (5 + E[X]) + (7 + E[X])\right) \\
&= 5 + \frac{2}{3}E[X].
\end{aligned}$$

Solving for $E[X]$, we get

$$E[X] = 15.$$

So

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\begin{aligned}
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Example 6

Suppose that the number of people entering a store on a given day is a random variable with mean 50. Suppose further that the amount of money spent by these customers are independent random variables with a common mean \$8. Assume also that the amount of money spent by a customer is also independent of the total number of people to enter the store. Find the expected amount of money spent in the store on a given day.

Let N be the number of people that enter the store on a given day. For $i = 1, 2, \ldots,$ let X_i be the amount of money spent by the i-th customer. Then the total amount of money spent in the store is $\sum_{i=1}^{N} X_i$.

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Let N be the number of people that enter the store on a given day. For $i = 1, 2, \ldots$, let X_i be the amount of money spent by the i-th customer. Then the total amount of money spent in the store is $\sum_{i=1}^{N} X_i$.

$$E[\sum_{i=1}^{N} X_i] = E[E[\sum_{i=1}^{N} X_i|N]].$$

$$E[\sum_{i=1}^{N} X_i|N=n] = E[\sum_{i=1}^{n} X_i|N=n] = E[\sum_{i=1}^{n} X_i] = 8n.$$

Thus

$$E[\sum_{i=1}^{N} X_i|N] = 8N$$

and

$$E[\sum_{i=1}^{N} X_i] = E[8N] = 8E[N] = 400.$$

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