Math 461 Fall 2021

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Outline

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Solution to HW8 will be on my homepage later this afternoon.

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Theorem

If X and Y are independent random variables, then for any functions ϕ and ψ on $\mathbb{R}$,

$$E[\phi(X)\psi(Y)] = E[\phi(X)]E[\psi(Y)].$$

Let's prove this in the absolute continuous case. Let f_X and f_Y be the density of X and Y respectively. Since X and Y are independent, the joint density of X and Y is $f_X(x)f_Y(y)$. Thus

$$E[\phi(X)\psi(Y)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \phi(x)\psi(y)f_X(x)f_Y(y)dxdy$$

$$= \int_{-\infty}^{\infty} \phi(x)f_X(x)dx \int_{-\infty}^{\infty} \psi(y)f_Y(y)dy = E[\phi(X)]E[\psi(Y)].$$

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Definition

The covariance $\text{Cov}(X, Y)$ of two random variables X and Y is defined by

$$\text{Cov}(X, Y) = E[(X - E[X])(Y - E[Y])].$$

One can easily check that

$$\text{Cov}(X, Y) = E[XY] - E[X]E[Y].$$

If X and Y are independent, then $\text{Cov}(X, Y) = 0$. But the converse is not true. When $\text{Cov}(X, Y) = 0$, we say that X and Y are uncorrelated. Independence implies uncorrelated, but not the other way around.

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Example 1

Suppose $P(X = 0) = P(X = 1) = P(X = -1) = \frac{1}{3}$ and

$$Y = \begin{cases} 0, & \text{if } X \neq 0, \\ 1, & \text{if } X = 0. \end{cases}$$

Then $XY = 0$, thus $E[XY] = 0$. We also have $E[X] = 0$ and $E[Y] = \frac{1}{3}$, thus $E[X]E[Y] = 0$. Hence $\text{Cov}(X, Y) = 0$. But X and Y are obviously not independent.

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Then $XY=0$, thus $E[XY]=0$. We also have $E[X]=0$ and $E[Y]=\frac{1}{3}$, thus $E[X]E[Y]=0$. Hence $\text{Cov}(X,Y)=0$. But X and Y are obviously not independent.

Example 2

A box has 3 balls labeled 1, 2, 3. Two Balls are randomly selected without replacement. Let X be the number on the first ball and Y the number on the second. Find $\text{Cov}(X, Y)$.

$$\begin{aligned} & P(X=1, Y=2) = P(X=1, Y=3) = P(X=2, Y=1) \\ = & P(X=2, Y=3) = P(X=3, Y=1) = P(X=3, Y=2) = \frac{1}{6}. \end{aligned}$$

Thus

$$E[XY] = (2+3+2+3+6+6)\frac{1}{6} = \frac{11}{3}.$$

Since $P(X=1) = P(X=2) = P(X=3) = P(Y=1) = P(Y=2) = P(Y=3) = \frac{1}{3}$, $E[X] = E[Y] = 2$. Hence $\text{Cov}(X, Y) = -\frac{1}{3}$.

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Example 3

The joint density of X and Y is given by

$$f(x,y) = \begin{cases} x+y, & 0<x<1, 0<y<1, \\ 0, & \text{otherwise.} \end{cases}$$

Find $\mathrm{Cov}(X, Y)$.

$$E[XY] = \int_0^1 \int_0^1 xy(x+y)dxdy = \int_0^1 \int_0^1 (x^2y + xy^2)dxdy$$
$$= \int_0^1 (\frac{y}{3} + \frac{y^2}{2})dy = \frac{1}{3}.$$

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Similarly, $E[Y] = \frac{7}{12}$.

Thus

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Proposition

- **(i)** $\mathrm{Cov}(X, Y) = \mathrm{Cov}(Y, X)$.
- **(ii)** $\mathrm{Cov}(X, X) = \mathrm{Var}(X)$.
- **(iii)** $\mathrm{Cov}(aX, Y) = a\mathrm{Cov}(X, Y)$.
- **(iv)** $\mathrm{Cov}(\sum_{i=1}^m X_i, \sum_{j=1}^n Y_j) = \sum_{i=1}^m \sum_{j=1}^n \mathrm{Cov}(X_i, Y_j)$.

(i), (ii) and (iii) are obvious. Let's look at $\mathrm{Cov}(X_1 + X_2, Y)$.

$$\begin{aligned}\mathrm{Cov}(X_1 + X_2, Y) &= E[(X_1 + X_2)Y] - E[X_1 + X_2]E[Y] \\ &= E[X_1 Y] - E[X_1]E[Y] + E[X_2 Y] - E[X_2]E[Y] \\ &= \mathrm{Cov}(X_1, Y) + \mathrm{Cov}(X_2, Y).\end{aligned}$$

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The general case follows by induction.

Combining (ii) and (iv) above, we get

$$\begin{aligned}\mathrm{Var}\Big(\sum_{i=1}^{n} X_i\Big) &= \sum_{i=1}^{n} \mathrm{Var}(X_i) + \sum_{i\neq j} \mathrm{Cov}(X_i, X_j)\\ &= \sum_{i=1}^{n} \mathrm{Var}(X_i) + 2\sum_{i<j} \mathrm{Cov}(X_i, X_j).\end{aligned}$$

In particular, if $X_1, \ldots, X_n$ are independent, then

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Example 4

Suppose that X_1, X_2, X_3 are independent, $\mathrm{Var}(X_1) = \sigma_1^2$, $\mathrm{Var}(X_2) = \sigma_2^2$ and $\mathrm{Var}(X_3) = \sigma_3^2$. Find $\mathrm{Cov}(X_1 - X_2, X_2 + X_3)$.

$$\mathrm{Cov}(X_1 - X_2, X_2 + X_3) = \mathrm{Cov}(X_1, X_2 + X_3) - \mathrm{Cov}(X_2, X_2 + X_3)$$
$$= -\mathrm{Cov}(X_2, X_2 + X_3) = -\mathrm{Cov}(X_2, X_2) - \mathrm{Cov}(X_2, X_3)$$
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Example 5

Let $X_1, \ldots, X_n$ be independent and identically distributed random variables with common mean μ and common variance σ^2.

$$\overline{X} = \frac{1}{n} \sum_{i=1}^{n} X_i$$

is called the sample mean and

$$S^2 = \frac{1}{n-1} \sum_{i=1}^{n} (X_i - \overline{X})^2$$

is called the sample variance. Find (a) $\mathrm{Var}(\overline{X})$; (b) $E[S^2]$.

$$\mathrm{Var}(\overline{X}) = \frac{1}{n^2} \mathrm{Var}\left(\sum_{i=1}^{n} X_i\right) = \frac{\sigma^2}{n}.$$

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For (b), we start with the following algebraic identity

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\begin{aligned}
(n-1)S^2 &= \sum_{i=1}^{n}(X_i - \mu + \mu - \overline{X})^2 \\
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&= \sum_{i=1}^{n}(X_i - \mu)^2 + n(\overline{X} - \mu)^2 - 2(\overline{X} - \mu)n(\overline{X} - \mu) \\
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Taking expectation, we get

$$(n-1)E[S^2] = n\sigma^2 - n\mathrm{Var}(\overline{X}) = (n-1)\sigma^2,$$

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since $E[\overline{X}] = \mu$. Hence $E[S^2] = \sigma^2$.

Definition

The correlation coefficient $\rho(X, Y)$ of two random variables X and Y is defined by

$$\rho(X, Y) = \frac{\mathrm{Cov}(X, Y)}{\sqrt{\mathrm{Var}(X)\mathrm{Var}(Y)}}.$$

We always have $|\rho(X, Y)| \leq 1$. $\rho(X, aX) = 1$ if $a > 0$, $\rho(X, aX) = -1$ if $a < 0$, and $\rho(X, Y) = 0$ if X and Y are independent. $|\rho(X, Y)| = 1$ if and only if $P(X = aY) = 1$ for some $a \neq 0$.

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Now we are going to use the formula

$$\begin{aligned}\operatorname{Var}\Big(\sum_{i=1}^n X_i\Big) &= \sum_{i=1}^n \operatorname{Var}(X_i) + \sum_{i\neq j} \operatorname{Cov}(X_i, X_j) \\ &= \sum_{i=1}^n \operatorname{Var}(X_i) + 2\sum_{i<j} \operatorname{Cov}(X_i, X_j).\end{aligned}$$

to find the variance of some complicated random variables.

Example 6

Suppose S_n is a binomial random variable with parameters (n, p). S_n is the total number of successes in n indep trails each of which results in a success with probability p. For $i = 1, \ldots, n$, let $X_i = 1$ if the i-th trial results in a success and $X_i = 0$ otherwise. Then $X_1, \ldots, X_n$ indep Bernoulli random variables with parameter p and $S_n = \sum_{i=1}^n X_i$. Thus

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Example 7

Let X be a negative binomial random variable with parameters (r, p). X is the number of trials needed in order to get r successes. Let Y_1 be the number of trials needed in order to get the first success; let Y_2 be the number of additional trials, after the first success, to get the second success, ..., let Y_r be the number of additional trials, after the $(r-1)$-st success, to get the r-th success. Then $Y_1, \ldots, Y_r$ are independent geometric random variables with parameter p and $X = Y_1 + \cdots + Y_r$. Thus

$$\mathrm{Var}(X) = \sum_{i=1}^{r} \mathrm{Var}(Y_i) = \frac{r(1-p)}{p^2}.$$