Math 461 Fall 2021

Renming Song

University of Illinois at Urbana-Champaign

October 20, 2021

Outline

Outline

HW7 is due Friday, 10/22, before the end of class time . Please submit your HW7 via the course Moodle page.

Solution to HW6 is on my homepage now.

HW7 is due Friday, 10/22, before the end of class time . Please submit your HW7 via the course Moodle page.

Solution to HW6 is on my homepage now.

Outline

Two random variables X and Y are said to be independent if for any two subsets A and B of $\mathbb{R}$,

$$P(X \in A, Y \in B) = P(X \in A)P(Y \in B).$$

It can be shown that X and Y are independent if and only if

$$F(x, y) = F_X(x)F_Y(y), \quad (x, y) \in \mathbb{R}^2.$$

Two random variables X and Y are said to be independent if for any two subsets A and B of $\mathbb{R}$,

$$P(X \in A, Y \in B) = P(X \in A)P(Y \in B).$$

It can be shown that X and Y are independent if and only if

$$F(x, y) = F_X(x)F_Y(y), \quad (x, y) \in \mathbb{R}^2.$$

It can be shown if X and Y are discrete random variables with joint mass function $p(\cdot,\cdot)$, then X and Y are independent if and only if

$$p(x,y) = p_X(x)p_Y(y), \quad (x,y) \in \mathbb{R}^2.$$

It can be shown if X and Y are jointly absolutely continuous with joint density $f(\cdot,\cdot)$, then X and Y are independent if and only if

$$f(x,y) = f_X(x)f_Y(y), \quad (x,y) \in \mathbb{R}^2.$$

It can be shown if X and Y are discrete random variables with joint mass function $p(\cdot,\cdot)$, then X and Y are independent if and only if

$$p(x,y) = p_X(x)p_Y(y), \quad (x,y) \in \mathbb{R}^2.$$

It can be shown if X and Y are jointly absolutely continuous with joint density $f(\cdot,\cdot)$, then X and Y are independent if and only if

$$f(x,y) = f_X(x)f_Y(y), \quad (x,y) \in \mathbb{R}^2.$$

Example 2

Suppose that the number of people entering a certain post office on a given day is a Poisson random variable with parameter $\lambda > 0$. Assume that each person entering the post office is male with probability p and female with probability $1 - p$, independent of all others. Show that the number of males and the number of females entering the post office on a given day are independent Poisson random variables with parameters λp and $\lambda(1 - p)$ respectively.

Let X and Y be the number of males and the number of females entering the post office on a given day respectively. X and Y are non-negative integer-valued random variables.

Example 2

Suppose that the number of people entering a certain post office on a given day is a Poisson random variable with parameter $\lambda > 0$. Assume that each person entering the post office is male with probability p and female with probability $1 - p$, independent of all others. Show that the number of males and the number of females entering the post office on a given day are independent Poisson random variables with parameters λp and $\lambda(1 - p)$ respectively.

Let X and Y be the number of males and the number of females entering the post office on a given day respectively. X and Y are non-negative integer-valued random variables.

For any non-negative integers i and j,

$$P(X = i, Y = j) = P(X = i, Y = j | X + Y = i + j)P(X + Y = i + j)$$
$$= \binom{i+j}{i} p^i (1-p)^j e^{-\lambda} \frac{\lambda^{i+j}}{(i+j)!}$$
$$= e^{-\lambda p} \frac{(\lambda p)^i}{i!} e^{-\lambda(1-p)} \frac{(\lambda(1-p))^j}{j!}.$$

Hence

$$P(X = i) = e^{-\lambda p} \frac{(\lambda p)^i}{i!} \sum_{j=0}^{\infty} e^{-\lambda(1-p)} \frac{(\lambda(1-p))^j}{j!} = e^{-\lambda p} \frac{(\lambda p)^i}{i!}.$$

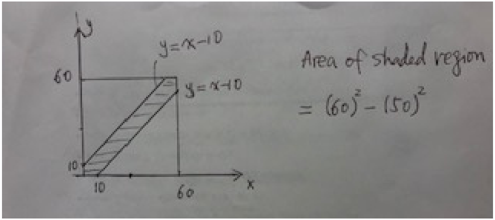
Similarly

$$P(Y = j) = e^{-\lambda(1-p)} \frac{(\lambda(1-p))^j}{j!}.$$

Therefore X and Y are independent Poisson random variables with parameters λp and $\lambda(1-p)$ respectively.

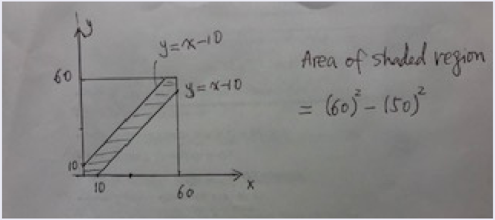
Example 3

A man and a woman decide to meet at a certain location. If each of them independently arrives at a time uniformly distributed between noon and 1 pm. Find the probability that the first to arrive needs to wait no more than 10 minutes.



Example 3

A man and a woman decide to meet at a certain location. If each of them independently arrives at a time uniformly distributed between noon and 1 pm. Find the probability that the first to arrive needs to wait no more than 10 minutes.



The answer is

$$\frac{60^2 - 50^2}{60^2} = \frac{11}{36}.$$

Proposition

(i) Suppose that X and Y are discrete with joint mass function $p(\cdot,\cdot)$. Then X and Y are independent if and only if

$$p(x,y) = g(x)h(y), \quad (x,y) \in \mathbb{R}^2$$

for some functions g and h on $\mathbb{R}$.

(ii) Suppose that X and Y are jointly abs. cont. with joint density $f(\cdot,\cdot)$. Then X and Y are independent if and only if

$$f(x,y) = g(x)h(y), \quad (x,y) \in \mathbb{R}^2,$$

for some functions g and h on $\mathbb{R}$.

The answer is

$$\frac{60^2 - 50^2}{60^2} = \frac{11}{36}.$$

Proposition

(i) Suppose that X and Y are discrete with joint mass function $p(\cdot,\cdot)$. Then X and Y are independent if and only if

$$p(x,y) = g(x)h(y), \quad (x,y) \in \mathbb{R}^2$$

for some functions g and h on $\mathbb{R}$.

(ii) Suppose that X and Y are jointly abs. cont. with joint density $f(\cdot,\cdot)$. Then X and Y are independent if and only if

$$f(x,y) = g(x)h(y), \quad (x,y) \in \mathbb{R}^2,$$

for some functions g and h on $\mathbb{R}$.

Example 4

The joint density of X and Y is

$$f(x,y) = \begin{cases} 10e^{-(2x+5y)}, & x > 0, y > 0 \\ 0, & \text{otherwise.} \end{cases}$$

If

$$g(x) = \begin{cases} 10e^{-2x}, & x > 0, \\ 0, & \text{otherwise,} \end{cases} \qquad h(y) = \begin{cases} e^{-5y}, & y > 0, \\ 0, & \text{otherwise.} \end{cases}$$

Then

$$f(x,y) = g(x)h(y), \quad (x,y) \in \mathbb{R}^2.$$

Thus X and Y are independent.

Example 4

The joint density of X and Y is

$$f(x,y) = \begin{cases} 10e^{-(2x+5y)}, & x > 0, y > 0 \\ 0, & \text{otherwise.} \end{cases}$$

If

$$g(x) = \begin{cases} 10e^{-2x}, & x > 0, \\ 0, & \text{otherwise,} \end{cases} \qquad h(y) = \begin{cases} e^{-5y}, & y > 0, \\ 0, & \text{otherwise.} \end{cases}$$

Then

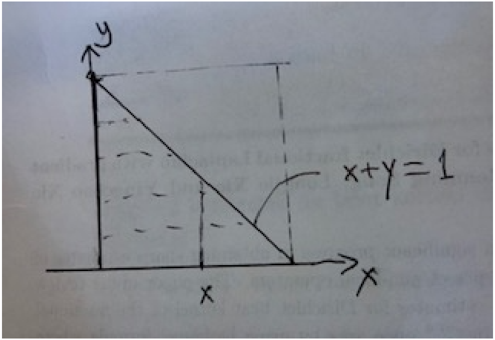
$$f(x,y) = g(x)h(y), \quad (x,y) \in \mathbb{R}^2.$$

Thus X and Y are independent.

Example 5

The joint density of X and Y is

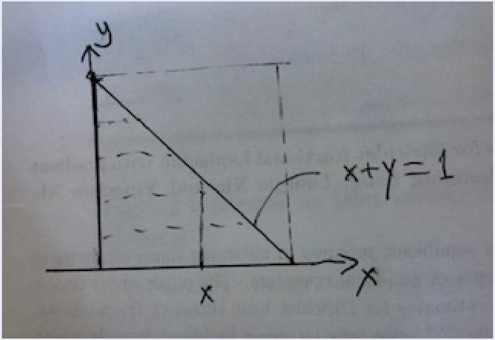
$$f(x,y) = \begin{cases} 24xy, & x \in (0,1), y \in (0,1), x+y \in (0,1) \\ 0, & \text{otherwise.} \end{cases}$$



Example 5

The joint density of X and Y is

$$f(x,y) = \begin{cases} 24xy, & x \in (0,1), y \in (0,1), x+y \in (0,1) \\ 0, & \text{otherwise.} \end{cases}$$



Both X and Y take values in $(0, 1)$. For $x \in (0, 1)$,

$$f_X(x) = \int_0^{1-x} 24xydy = 12x(1-x)^2.$$

Similarly, for $y \in (0, 1)$,

$$f_Y(y) = 12y(1-y)^2.$$

X and Y are not independent!

The concept of independent random variables can be extended to more than 2 random variables.

Both X and Y take values in $(0,1)$. For $x \in (0,1)$,

$$f_X(x) = \int_0^{1-x} 24xydy = 12x(1-x)^2.$$

Similarly, for $y \in (0,1)$,

$$f_Y(y) = 12y(1-y)^2.$$

X and Y are not independent!

The concept of independent random variables can be extended to more than 2 random variables.

n random variables $X_1, \ldots, X_n$ are said to be independent if for any subsets $A_1, \ldots, A_n$ of $\mathbb{R}$,

$$P(X_1 \in A_1, \ldots X_n \in A_n) = \prod_{i=1}^{n} P(X_i \in A_i).$$

It can be shown that n random variables $X_1, \ldots, X_n$ with joint distribution function $F(\cdot, \ldots, \cdot)$ are independent if and only if

$$F(x_1, \ldots, x_n) = \prod_{i=1}^{n} F_{X_i}(x_i), \quad (x_1, \ldots, x_n) \in \mathbb{R}^n.$$

n random variables $X_1, \ldots, X_n$ are said to be independent if for any subsets $A_1, \ldots, A_n$ of $\mathbb{R}$,

$$P(X_1 \in A_1, \ldots X_n \in A_n) = \prod_{i=1}^{n} P(X_i \in A_i).$$

It can be shown that n random variables $X_1, \ldots, X_n$ with joint distribution function $F(\cdot, \ldots, \cdot)$ are independent if and only if

$$F(x_1, \ldots, x_n) = \prod_{i=1}^{n} F_{X_i}(x_i), \quad (x_1, \ldots, x_n) \in \mathbb{R}^n.$$

It can be shown that n discrete random variables $X_1, \dots, X_n$ with joint mass function $p(\cdot, \dots, \cdot)$ are independent if and only if

$$p(x_1, \dots, x_n) = \prod_{i=1}^{n} p_{X_i}(x_i), \quad (x_1, \dots, x_n) \in \mathbb{R}^n.$$

It can be shown that n jointly abs cont. random variables $X_1, \dots, X_n$ with joint density $f(\cdot, \dots, \cdot)$ are independent if and only if

$$f(x_1, \dots, x_n) = \prod_{i=1}^{n} f_{X_i}(x_i), \quad (x_1, \dots, x_n) \in \mathbb{R}^n.$$

It can be shown that n discrete random variables $X_1, \dots, X_n$ with joint mass function $p(\cdot, \dots, \cdot)$ are independent if and only if

$$p(x_1, \dots, x_n) = \prod_{i=1}^{n} p_{X_i}(x_i), \quad (x_1, \dots, x_n) \in \mathbb{R}^n.$$

It can be shown that n jointly abs cont. random variables $X_1, \dots, X_n$ with joint density $f(\cdot, \dots, \cdot)$ are independent if and only if

$$f(x_1, \dots, x_n) = \prod_{i=1}^{n} f_{X_i}(x_i), \quad (x_1, \dots, x_n) \in \mathbb{R}^n.$$

Example 6

Suppose that $X_1, \ldots, X_n$ are independent absolutely continuous random random variables with a common density f. . Define

$$U = \min\{X_1, \ldots, X_n\}, \quad V = \max\{X_1, \ldots, X_n\}.$$

Find the densities of U and V respectively.

Let's deal with V first. Let F be the common distribution. For any $v \in \mathbb{R}$,

$$P(V \leq v) = P(X_1 \leq v, \ldots, X_n \leq v) = (F(v))^n.$$

Thus the density of V is $f_V(v) = n(F(v))^{n-1} f(v)$.

Example 6

Suppose that $X_1, \ldots, X_n$ are independent absolutely continuous random random variables with a common density f. . Define

$$U = \min\{X_1, \ldots, X_n\}, \quad V = \max\{X_1, \ldots, X_n\}.$$

Find the densities of U and V respectively.

Let's deal with V first. Let F be the common distribution. For any $v \in \mathbb{R}$,

$$P(V \le v) = P(X_1 \le v, \ldots, X_n \le v) = (F(v))^n.$$

Thus the density of V is $f_V(v) = n(F(v))^{n-1} f(v)$.

Now let's deal with U. For any $u \in \mathbb{R}$,

$$\begin{aligned} P(U \le u) &= 1 - P(U > u) = 1 - P(X_1 > u, \ldots, X_n > u) \\ &= 1 - (1 - F(u))^n. \end{aligned}$$

Thus the density of U is

$$f_U(u) = n(1 - F(u))^{n-1} f(u).$$

We can also find the joint density of U and V. I will come back to this later in this chapter.

Now let's deal with U. For any $u \in \mathbb{R}$,

$$\begin{aligned} P(U \leq u) &= 1 - P(U > u) = 1 - P(X_1 > u, \ldots, X_n > u) \\ &= 1 - (1 - F(u))^n. \end{aligned}$$

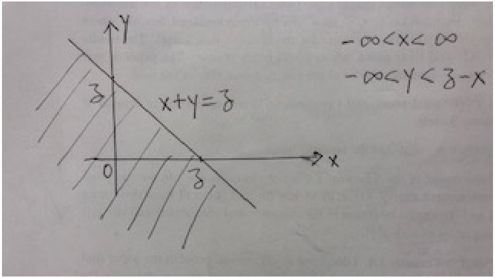
Thus the density of U is

$$f_U(u) = n(1 - F(u))^{n-1} f(u).$$

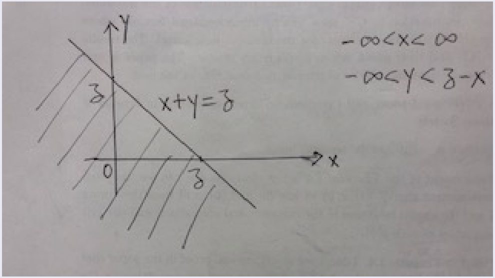
We can also find the joint density of U and V. I will come back to this later in this chapter.

Outline

Suppose X and Y are independent abs. cont. random variables with density f_X and f_Y respectively. Find the density of $Z = X + Y$.



Suppose X and Y are independent abs. cont. random variables with density f_X and f_Y respectively. Find the density of $Z = X + Y$.



For any $z \in \mathbb{R}$,

$$\begin{aligned} F_Z(z) &= P(X+Y \le z) \\ &= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{z-x} f_X(x) f_Y(y) dy \right) dx \\ &= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{z} f_X(x) f_Y(v-x) dv \right) dx, \quad (y = v - x) \\ &= \int_{-\infty}^{z} \int_{-\infty}^{\infty} f_X(x) f_Y(v-x) dx dv \end{aligned}$$

Thus the density of Z is

$$f_Z(z) = \int_{-\infty}^{\infty} f_X(x) f_Y(z-x) dx.$$

Similarly, we also have

$$f_Z(z) = \int_{-\infty}^{\infty} f_X(z-y) f_Y(y) dy.$$

For any $z \in \mathbb{R}$,

$$\begin{aligned} F_Z(z) &= P(X+Y \le z) \\ &= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{z-x} f_X(x) f_Y(y) dy \right) dx \\ &= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{z} f_X(x) f_Y(v-x) dv \right) dx, \quad (y = v - x) \\ &= \int_{-\infty}^{z} \int_{-\infty}^{\infty} f_X(x) f_Y(v-x) dx dv \end{aligned}$$

Thus the density of Z is

$$f_Z(z) = \int_{-\infty}^{\infty} f_X(x) f_Y(z-x) dx.$$

Similarly, we also have

$$f_Z(z) = \int_{-\infty}^{\infty} f_X(z-y) f_Y(y) dy.$$