Math 461 Fall 2021

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University of Illinois at Urbana-Champaign

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Outline

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HW6 is due today before the end of the class. You can either submit your HW6 via the course Moodle page, or submit a hard copy.

Solution to Test 1 is on my homepage.

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Example 3

Suppose that X is uniformly distributed in $(-\pi/2, \pi/2)$. Find the density of $Y = \tan X$.

For any real number y,

$$P(Y \leq y) = P(\tan X \leq y) = P(X \leq \arctan y) = \frac{1}{2} + \frac{1}{\pi} \arctan y.$$

Thus the density of Y is

$$f_Y(y) = \frac{1}{\pi(1+y^2)}.$$

Y is a Cauchy random variable.

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If X is a continuous random variable with distribution function F, then $Y = F(X)$ is uniformly distributed on $(0, 1)$.

Let F be a continuous distribution function that is strictly increasing on some interval I and such that $F = 0$ to the left of I (if I is bounded from below) and $F = 1$ to the right of I (if I is bounded from above). Thus $F^{-1}(y)$, $0 < y < 1$, is well defined. If U is uniformly distributed on $(0, 1)$, then $X = F^{-1}(U)$ is a random variable with distribution function F.

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So far we have been dealing with distribution functions for single random variables. However, we are often interested in probability statements concerning more than one random variable. In order to deal with such probabilities, we define the joint (cumulative) distribution functions of random variables.

The joint distribution function of two random variables X and Y is defined as

$$F(x, y) = P(X \leq x, Y \leq y), \quad -\infty < x, y < \infty.$$

It is a function on the plane.

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It is a function on the plane.

If we know the joint distribution F of X and Y, then we can find the distributions F_X and F_Y of X and Y easily.

$$\begin{aligned} F_X(x) &= P(X \leq x) = P(X \leq x, Y < \infty) \\ &= P(\lim_{y\to\infty}\{X \leq x, Y \leq y\}) = \lim_{y\to\infty} P(X \leq x, Y \leq y) \\ &= \lim_{y\to\infty} F(x, y) = F(x, \infty). \end{aligned}$$

Similarly,

$$F_Y(y) = \lim_{x\to\infty} F(x, y) = F(\infty, y).$$

The distribution functions F_X and F_Y are called the marginal distributions of X and Y respectively. In general, knowing the marginal distributions is not enough to recover the joint distribution function.

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The distribution functions F_X and F_Y are called the marginal distributions of X and Y respectively. In general, knowing the marginal distributions is not enough to recover the joint distribution function.

The joint distribution function contains all the statistical information about X and Y.

Example 1

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$$\begin{aligned} & P(X > x, Y > y) = 1 - P(\{X > x, Y > y\}^c) \\ =& 1 - P(\{X > x\}^c \cup \{Y > y\}^c) = 1 - P(\{X \leq x\} \cup \{Y \leq y\}) \\ &= 1 - [P(X \leq x) + P(Y \leq y) - P(X \leq x, Y \leq y)] \\ &= 1 - F_X(x) - F_Y(y) + F(x, y). \end{aligned}$$

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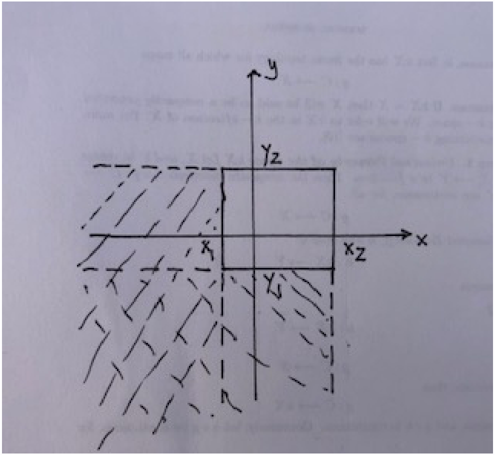
In general

$$P(x_1 < X \leq x_2, y_1 < Y \leq y_2)$$
$$= F(x_2, y_2) - F(x_1, y_2) - F(x_2, y_1) + F(x_1, y_1).$$



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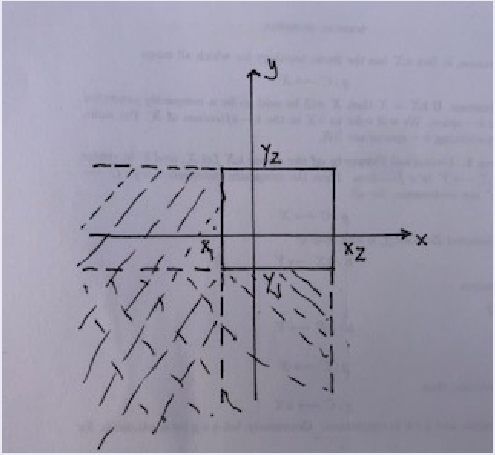

In the case when X and Y are both discrete random variables, it is more convenient to use the joint mass function of X and Y defined by

$$p(x, y) = P(X = x, Y = y), \quad \infty < x, y < \infty.$$

The mass functions p_X and p_Y of X and Y are called the marginal mass functions of X and Y respectively, and they can be determined by the joint mass function:

$$p_X(x) = P(X = x) = \sum_y P(X = x, Y = y) = \sum_y p(x, y)$$

$$p_Y(y) = P(Y = y) = \sum_x P(X = x, Y = y) = \sum_x p(x, y).$$

However, the joint mass function is not determined by the the marginal mass functions in general.

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Example 1

2 balls are randomly drawn, without replacement, from a box containing 3 balls labeled 1, 2 and 3. Let X be the number on the first ball and Y the number on the second ball. Find the joint mass function of X and Y.

$$p(1,2) = p(1,3) = p(2,1) = p(2,3) = p(3,1) = p(3,2) = \frac{1}{6}.$$

It is illustrative to record the joint mass function in the following table:

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It is illustrative to record the joint mass function in the following table:

X \ Y	1	2	3
1	0	1/6	1/6
2	1/6	0	$\frac{1}{6}$
3	$\frac{1}{6}$	$\frac{1}{6}$	0

You can read off the marginal mass functions p_X and p_Y easily. You can get p_X from the row sums, and p_Y from the column sums.

$X \backslash Y$	1	2	3
1	0	$1/6$	$1/6$
2	$1/6$	0	$\frac{1}{6}$
3	$\frac{1}{6}$	$\frac{1}{6}$	0

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Example 2

The joint mass function of X and Y is given by

$X \backslash Y$	-1	0	2	6
-2	$\frac{1}{9}$	$\frac{1}{27}$	$\frac{1}{27}$	$\frac{1}{9}$
1	$\frac{2}{9}$	0	$\frac{1}{9}$	$\frac{1}{9}$
3	0	0	$\frac{1}{9}$	$\frac{4}{27}$

Find the probability that (a) Y is even; (b) XY is odd; (c) $X > 0$ and $Y \geq 0$.

(a) $P(Y \text{ is even }) = P(Y = 0) + P(Y = 2) + P(Y = 6) = \frac{2}{3}$.
(b) $P(XY \text{ is odd }) = P(X = 1, Y = -1) + P(X = 3, Y = -1) = \frac{2}{9}$.
(c) $P(X > 0, Y \geq 0) = \frac{13}{27}$.

We say that 2 random variables X and Y are jointly absolutely continuous if there is a non-negative function f on $\mathbb{R}^2$ such that for any $x, y \in \mathbb{R}$,

$$P(X \leq x, Y \leq y) = \int_{-\infty}^{x} \left(\int_{-\infty}^{y} f(u, v) dv \right) du.$$

f is called the joint density of X and Y. f must satisfy

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) dxdy = 1.$$

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Conversely, any non-negative function f on $\mathbb{R}^2$ satisfying the condition above is the joint density of some random variables X and Y. More generally, any non-negative function g on $\mathbb{R}^2$ such that

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y)dxdy \in (0,\infty)$$

can be normalized to a joint density.

The joint density of X and Y contains all the statistical information about X and Y. For any region C in $\mathbb{R}^2$,

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If X and Y are jointly absolutely continuous with joint density f, then X and Y are both absolutely continuous, and the densities f_X and f_Y of X and Y are called the marginal densities of X and Y respectively.

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)dy, \quad f_Y(y) = \int_{-\infty}^{\infty} f(x,y)dx.$$

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Example 3

The joint density of X and Y is given by

$$f(x,y) = \begin{cases} 6e^{-2x}e^{-3y}, & x > 0, y > 0 \\ 0, & \text{otherwise.} \end{cases}$$

Find (a) $P(X > 1, Y < 1)$; (b) $P(X < Y)$.

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$$P(X > 1, Y < 1) = \int_1^\infty \left(\int_0^1 6e^{-2x}e^{-3y}\,dy \right) dx$$

$$= \int_1^\infty 2e^{-2x}\,dx \int_0^1 3e^{-3y}\,dy = e^{-2}(1 - e^{-3})$$

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$$= \int_0^\infty e^{-5x} dx = \frac{2}{5}.$$