Math 461 Fall 2021

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University of Illinois at Urbana-Champaign

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Outline

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HW6 is due Friday, 10/15, before the end of the class. You can either submit your HW6 via the course Moodle page, or submit a hard copy.

Solution to Test 1 is on my homepage. The distribution of the score of Test 1 is available on my homepage.

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For any $\lambda > 0$, the function

$$f(x) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0 \\ 0, & x < 0, \end{cases}$$

is a probability density. It is called an exponential density with parameter λ.

A random variable X is called an exponential random variable with parameter $\lambda > 0$ if it is an absolutely continuous random variable whose density is an exponential density with parameter λ.

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If X is an exponential random variable with parameter $\lambda > 0$, then for any $x \geq 0$,

$$P(X \leq x) = \int_0^x \lambda e^{-\lambda t} dt = 1 - e^{-\lambda x},$$

$$P(X > x) = e^{-\lambda x}.$$

Thus the distribution function of X is

$$F(x) = \begin{cases} 1 - e^{-\lambda x}, & x \geq 0 \\ 0, & x < 0. \end{cases}$$

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$$E[X] = \frac{1}{\lambda}, \quad \mathrm{Var}(X) = \frac{1}{\lambda^2}.$$

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$$E[X] = \int_0^\infty x\lambda e^{-\lambda x} dx = \int_0^\infty x d(-e^{-\lambda x})$$
$$= - xe^{-\lambda x}|_0^\infty + \int_0^\infty e^{-\lambda x} dx$$
$$= 0 - \frac{1}{\lambda} e^{-\lambda x}|_0^\infty = \frac{1}{\lambda},$$

For $n > 1$,

$$E[X^n] = \int_0^\infty x^n \lambda e^{-\lambda x} dx = \int_0^\infty x^n d(-e^{-\lambda x})$$
$$= - x^n e^{-\lambda x}|_0^\infty + \int_0^\infty n x^{n-1} e^{-\lambda x} dx$$
$$= \frac{n}{\lambda} \int_0^\infty x^{n-1} \lambda e^{-\lambda x} dx = \frac{n}{\lambda} E[X^{n-1}].$$

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Example

Suppose that the length X of a phone call in minutes is an exponential random variable with parameter $\lambda = 1/5$. Find the probability that the phone call will (a) last more than 5 minutes; (b) last between 5 and 10 minutes.

$$P(X > 5) = \int_5^\infty \frac{1}{5} e^{-x/5} dx = e^{-1}.$$

$$P(5 \leq X \leq 10) = \int_5^{10} \frac{1}{5} e^{-x/5} dx = e^{-1} - e^{-2}.$$

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Suppose X is an exponential random variable with parameter $\lambda > 0$. For any $s, t > 0$,

$$P(X > s + t | X > t) = \frac{P(X > s + t)}{P(X > t)}$$

$$= \frac{e^{-\lambda(s+t)}}{e^{-\lambda t}} = e^{-\lambda s} = P(X > s).$$

This property is called the memoryless property. Any exponential random variable satisfies the memoryless property.

The memoryless property is equivalent to

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It can be shown that if g is a non-negative right continuous function on $(0, \infty)$ taking values in $(0, 1)$ such that

$$g(s+t) = g(s)g(t), \quad s, t > 0,$$

then there exists $\lambda > 0$ such that

$$g(t) = e^{-\lambda t}, \quad t > 0.$$

Thus if a random variable satisfies the memoryless property, it must be an exponential random variable. Thus exponential random variables are very important in applications.

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If the lifetime of a certain object, like a light bulb or computer chip, has the memoryless property, then we can use the exponential distribution to model the lifetime. The lifetime of a car usually does not satisfy the memoryless property, thus it is not reasonable to use an exponential random variable to model the lifetime of a car.

Example 2

Suppose that X an exponential random variable with parameter $\lambda > 0$. Define a new random variable Y as follows: $Y = n$ when $X \in (n-1, n]$, $n = 1, 2, \ldots$. Find the mass function of Y.

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For $n = 1, 2, \ldots,$

$$P(Y = n) = P(n - 1 < X \leq n) = e^{-\lambda(n-1)} - e^{-\lambda n}$$
$$= e^{-\lambda(n-1)}(1 - e^{-\lambda}).$$

Thus Y is a geometric random variable with parameter $p = 1 - e^{-\lambda}$.

Geometric random variables are the discrete counterpart of exponential random variables. Exponential random variables are the continuous counterpart of geometric random variables.

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Example 3

Suppose that the lifetime X of a light bulb in months is an exponential random variable with parameter $\lambda = 1/12$. If the light bulb has been working for 12 months, find the probability that it will work for another 12 months.

By the meomoryless property

$$P(X > 24 | X > 12) = P(X > 12) = e^{-1}.$$

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For any $\alpha > 0$, we define

$$\Gamma(\alpha) = \int_0^\infty y^{\alpha-1} e^{-y} dy.$$

For any $\alpha > 0$, $\Gamma(\alpha) \in (0, \infty)$. But we do not know the value of $\Gamma(\alpha)$ in general. We do know that $\Gamma(1) = 1$.

We claim that, for any $\alpha > 0$, $\Gamma(\alpha + 1) = \alpha\Gamma(\alpha)$. Combining this with $\Gamma(1) = 1$, we immediately get $\Gamma(n) = (n - 1)!$ for all $n \geq 1$.

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$$\begin{aligned}\Gamma(\alpha+1) &= \int_0^\infty y^\alpha e^{-y} dy = \int_0^\infty y^\alpha d(-e^{-y}) \\ &= - y^\alpha e^{-y}\Big|_0^\infty + \int_0^\infty \alpha y^{\alpha-1} e^{-y} dy \\ &= \alpha\Gamma(\alpha).\end{aligned}$$

By using a simple change of variables, one can check that, for any $\alpha > 0$ and $\lambda > 0$, the function

$$f(x) = \begin{cases} \frac{1}{\Gamma(\alpha)}(\lambda x)^{\alpha-1}\lambda e^{-\lambda x}, & x \geq 0 \\ 0, & x < 0, \end{cases}$$

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