

# Math 461 Fall 2021

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# Outline

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- 1 **General Info**
- 2 4.9 Expectation of Sums of Random Variables
- 3 5.1 Introduction

HW4 is due today, before the end of class.. You can either submit a hard copy or electronically as a pdf file via the HW4 folder in the course Moodle page.

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### Example 5

$n$  balls are randomly distributed into  $r$  boxes (so that each ball is equally likely to go to any of the  $r$  boxes). Find the expected number of empty boxes.

Let  $X$  be the number of empty boxes. For  $i = 1, \dots, r$ , let  $X_i = 1$  if box number  $i$  is empty and  $X_i = 0$  otherwise. Then  $X = X_1 + \dots + X_r$ . Note that for  $i = 1, \dots, r$ ,

$$E[X_i] = P(X_i = 1) = \left(\frac{r-1}{r}\right)^n.$$

Thus

$$E[X] = E[X_1] + \dots + E[X_r] = r \left(\frac{r-1}{r}\right)^n.$$

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### Example 6

Suppose that there are  $N$  types of coupons and each time one gets a coupon it is equally likely to be any one of the  $N$  types. (a) If you have just collected  $n$  coupons, what is the expected number of different types of coupons in your collection? (b) Find the expected number of coupons you need to amass in order to get a complete set.

Let  $X$  be the number of different types in your collection of  $n$  coupons. For  $i = 1, \dots, N$ , let  $X_i = 1$  if there is at least one type  $i$  coupon in your collection and  $X_i = 0$  otherwise. Then  $X = X_1 + \dots + X_N$ . Note that, for  $i = 1, \dots, N$ ,

$$E[X_i] = P(X_i = 1) = 1 - \left(\frac{N-1}{N}\right)^n.$$

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$$E[X] = E[X_1] + \dots + E[X_N] = N \left(1 - \left(\frac{N-1}{N}\right)^n\right).$$

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(b) Let  $Y$  be the number of coupons you need to amass in order to get a complete set. Let  $Y_1 = 1$ ; let  $Y_2$  be the number of additional coupons needed in order to get a new type (i.e., different from the one you got); let  $Y_3$  be the number of additional coupons, after you have got two types, needed in order to get a new type (different from the two types you already got);  $\dots$ , let  $Y_N$  be the number of additional coupons, after you have got  $N - 1$  types, to get the final type. Then  $Y = Y_1 + \dots + Y_N$ .

$Y_2$  is a geometric random variable with parameter  $\frac{N-1}{N}$ ;  $Y_3$  is a geometric random variable with parameter  $\frac{N-2}{N}$ ,  $\dots$ ,  $Y_N$  is a geometric random variable with parameter  $\frac{1}{N}$ . Thus

$$E[Y] = E[Y_1] + \dots + E[Y_N] = 1 + \frac{N}{N-1} + \dots + N = N\left(1 + \frac{1}{2} + \dots + \frac{1}{N}\right).$$

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The distribution function  $F$  of a random variable  $X$  is defined by

$$F(x) = P(X \leq x), \quad x \in \mathbb{R}.$$

The distribution function  $F$  of any random variable  $X$  satisfies

- (i)  $F$  is non-decreasing;
- (ii)  $\lim_{x \rightarrow \infty} F(x) = 1$  and  $\lim_{x \rightarrow -\infty} F(x) = 0$ ;
- (iii)  $F$  is right-continuous, i.e.,  $F(x+) = F(x)$  for every  $x \in \mathbb{R}$ .

We know that

$$P(X < b) = F(b-).$$

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In general, the distribution function  $F$  of a random variable is not continuous.

A random variable  $X$  is said to be continuous if its distribution function is continuous, or equivalently,

$$P(X = x) = 0, \quad \text{for every } x \in \mathbb{R}.$$

If  $X$  is a continuous random variable, then for any  $a \in \mathbb{R}$ ,

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General random variables can be classified as discrete random variables; continuous random variables; or neither discrete nor continuous.

The random variables with distribution function

$$F(x) = \begin{cases} 0, & x < 0, \\ x/3, & 0 \leq x < 1, \\ x/2, & 1 \leq x < 2, \\ 1, & x \geq 2. \end{cases}$$

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In Chapter 4, we learned about discrete random variables. In this chapter, we will concentrate on continuous random variables.

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In Chapter 4, we learned about discrete random variables. In this chapter, we will concentrate on continuous random variables.

In fact, we will concentrate on a subclass of continuous random variables – the so-called absolutely continuous random variables.

### Definition

A non-negative function  $f$  on  $\mathbb{R}$  is called a probability density function if

$$\int_{-\infty}^{\infty} f(x) dx = 1.$$

If  $f$  is a probability density function, then the function  $F$  defined by

$$F(x) = \int_{-\infty}^x f(t) dt, \quad x \in \mathbb{R} \quad (1)$$

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## Definition

A random variable  $X$  is said to be absolutely continuous if there is a non-negative function  $f$  on  $\mathbb{R}$  such that

$$P(X \leq x) = \int_{-\infty}^x f(t) dt, \quad x \in \mathbb{R}.$$

$f$  must be a probability density and it is called the density of  $X$ .

The density function of an absolutely continuous random variable  $X$  contains all the statistical info about  $X$ . If we know the density  $f$ , we can get the distribution  $F$  of  $X$  via (1). If we know the the distribution  $F$  of an absolutely continuous random variable  $X$ , we can simply differentiate  $F$  to get the density  $f$ . For points where  $F$  is not differentiable, we simply let  $f$  equal 0 there.

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If we know the density  $f$  of an absolutely continuous random variable  $X$ , then

$$P(a < X < b) = P(a \leq X \leq b) = \int_a^b f(x) dx.$$

There exists continuous random variables that are not absolutely continuous. But we are not going to deal with these type of “pathological” random variables in this course.

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### Example 1

Choose a point at random from the disk  $B(0, 1)$ . Let  $X$  be the distance between the chosen point and the origin.

For any  $x \in [0, 1]$ ,

$$P(X \leq x) = \frac{\pi x^2}{\pi} = x^2.$$

Thus the distribution function of  $X$  is

$$F(x) = \begin{cases} 0, & x \leq 0, \\ x^2, & 0 \leq x \leq 1, \\ 1, & x > 1. \end{cases}$$

The density of  $X$  is

$$f(x) = \begin{cases} 2x, & 0 < x < 1, \\ 0, & \text{otherwise.} \end{cases}$$

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