Math 461 Fall 2021

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Outline

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HW4 is due Friday, 09/24, before the end of class. You can either submit a hard copy or electronically as a pdf file via the HW4 folder in the course Moodle page.

Solutions to HW3 is on my homepage.

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Suppose that independent trials, each results in a success with probability p and a failure with probability $1-p$, are performed until a total of r successes is accumulated. Let X be the number of trials needed. Then

$$P(X=n)=\binom{n-1}{r-1}p^r(1-p)^{n-r}, \quad n=r,r+1,\ldots.$$

This random variable X is called a negative binomial random variable with parameters (r,p).

If X is a negative binomial random variable with parameters (r,p), then

$$E[X]=\frac{r}{p}, \quad \mathrm{Var}(X)=\frac{r(1-p)}{p^2}.$$

Deriving these formula using the definition will be pretty complicated. One can think along the following line: Let X_1 be the number of trials needed for the 1st success; let X_2 be the number of additional trials needed for the 2nd success, ... Then $X=X_1+\cdots+X_r$.

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The problem of points (revisted)

If independent trials, each results in a success with probability p and a failure with probability $1-p$, are performed, what is the probability of r successes occur before m failures?

Let X be the number of trials needed to get r successes. Then the answer is equal to

$$\sum_{n=r}^{r+m-1} P(X=n) = \sum_{n=r}^{r+m-1} \binom{n-1}{r-1} p^r (1-p)^{n-r}.$$

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Example (The Banach Match problem)

A pipe-smoking mathematician carries, at all times, 2 match boxes, 1 in his left pocket and 1 in his right pocket. Each time he needs a match, he is equally likely to take it from either box. Consider the first moment when he finds that one of the boxes is empty. If it is assumed that both boxes initially contained N matches, what is the probability that exactly k matches are in the other box?

Let E be the event that "he first discovers the right box is empty and the left box has exactly k matches at that time".

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E occurs when the $(N+1)$-st choice of the right box is made at the $(N+1)+(N-k)$-th trial. Thus

$$P(E) = \binom{2N-k}{N}\left(\frac{1}{2}\right)^{2N-k+1}.$$

The answer is equal to

$$2P(E) = \binom{2N-k}{N}\left(\frac{1}{2}\right)^{2N-k}.$$

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Expectation is just the average. So the following result, which I have mentioned explicitly several times, is very intuitive.

Proposition

If $X_1, X_2, \ldots, X_n$ are random variables with finite expectation, then $X = X_1 + \cdots + X_n$ has finite expectation and

$$E[\sum_{i=1}^{n} X_i] = \sum_{i=1}^{n} E[X_i].$$

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In this section, I will use this to find the expectation for some complicated random variables.

Example 1

Suppose that X is a binomial random variable with parameters (n, p). Find the $E[X]$.

X is the total number of successes in n independent trials, where each trial results in a success with probability p and a failure with probability $1 - p$. For $i = 1, \ldots, n$, let $X_i = 1$ if the i-th trial results in a success and $X_i = 0$ otherwise. Then each X_i is a Bernoulli random variable with parameter p and $X = X_1 + \cdots + X_n$. Thus

$$E[X] = E[X_1] + \cdots + E[X_n] = np.$$

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Example 2

Suppose that X is a negative binomial random variable with parameters (r, p). Find $E[X]$.

Recall X is the number of trials needed to get a total of r successes when each trial results in a success with probability p and a failure with probability $1 - p$. Let X_1 be the number of trials needed for the 1st success; let X_2 be the number of additional trials needed, after the first success, to get the 2nd success, ... Then each X_i is a geometric random variable with parameter p and $X = X_1 + \cdots + X_r$. Thus

$$E[X] = E[X_1] + \cdots + E[X_r] = \frac{r}{p}.$$

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Example 3

If n balls are randomly selected, without replacement, from a box containing N ($N > n$) balls, of which m are white, find the expected number of white balls selected.

Let X be the number of white balls selected. For $i = 1, \ldots, n$, let $X_i = 1$ if the i-th selected ball is white and $X_i = 0$ otherwise. Then $X = X_1 + \cdots + X_n$. Note that

$$E[X_i] = P(X_i = 1) = \frac{m}{N}.$$

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We could have solved the problem above by decomposing X in another way.

For $i = 1, \ldots, m$, let $Y_i = 1$ if the i-th white ball is among the selected and $Y_i = 0$ otherwise. Then $X = Y_1 + \cdots + Y_m$. Note that

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Example 4

A group of N people throw their hats into the center of the room. The hats are mixed up, and each person randomly selects a hat. Let X be the number of people who get their own hats. Find $E[X]$.

For $i = 1, \ldots, N$, let $X_i = 1$ if the i-th man gets his own hat and $X_i = 0$ otherwise. Then $X = X_1 + \cdots + X_N$. Note that

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