Math 461 Fall 2021

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University of Illinois at Urbana-Champaign

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Outline

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1. General Info
2. 4.7 Poisson random variables
3. 4.8 Other Discrete Probability Distributions

HW4 is due Friday, 09/24, before the end of class. You can either submit a hard copy or electronically as a pdf file via the HW4 folder in the course Moodle page.

Solutions to HW3 is on my homepage.

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Poisson random variables also arise in situations where "incidents" occur at certain points in time, like earthquakes, people entering a certain establishment.

In a lot of situations, the following assumptions are (approximately) satisfied: For some $\lambda > 0$, the following hold:

1. The probability of exactly 1 incident occurs in a given interval of length h is $\lambda h + o(h)$,
2. The probability of 2 or more incidents occur in an interval of length h is $o(h)$.
3. For any integer $n \geq 1$, any non-negative integers $j_1, \ldots, j_n$, and any set of n non-overlapping intervals, if E_i denotes the event that exactly j_i incidents occur in the i-th interval, $i = 1, \ldots, n$, then $E_1, \ldots, E_n$ are independent.

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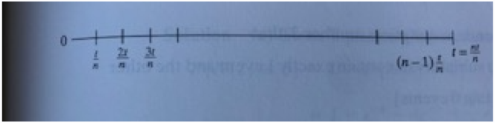
Under the assumptions above, the number of incidents occurring in any interval of length t is a Poisson random variable with parameter λt. It suffices to deal with the case when the interval is $[0, t]$.

Let $N(t)$ be the number of incidents occurring in $[0, t]$. For any $n \geq 1$, we divide $[0, t]$ into n sub-intervals of equal length:



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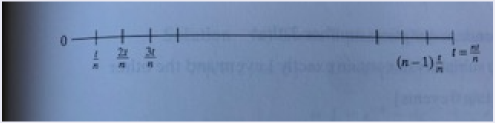

The event $\{N(t) = k\}$ can be written as the disjoint union of 2 events A and B where

A is the event that "k of the n sub-intervals contains exactly 1 incident each and the other $n-k$ sub-intervals contains 0 incident", and B is the event that "$N(t)=k$ and at least one of the sub-intervals contain 2 or more incidents". Thus $P(N(t)=k)=P(A)+P(B)$.

$$\begin{aligned} P(B) &\leq P(\text{at least 1 subinterval contain 2 or more incidents}) \\ &= P(\cup_{i=1}^{n}\{\text{ the } i\text{-th subinterval contain 2 or more incidents}) \\ &\leq \sum_{i=1}^{n} P(\text{ the } i\text{-th subinterval contain 2 or more incidents}) \\ &= \sum_{i=1}^{n} o(\frac{t}{n}) = n \cdot o(\frac{t}{n}) \to 0 \end{aligned}$$

as $n \to \infty$.

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$$P(A) = \binom{n}{k} \left(\frac{\lambda t}{n} + o(\frac{\lambda t}{n}) \right)^k \left(1 - \frac{\lambda t}{n} - o(\frac{\lambda t}{n}) \right)^{n-k}.$$

Since

$$n \left(\frac{\lambda t}{n} + o(\frac{\lambda t}{n}) \right) \to \lambda t,$$

we have

$$P(A) \to e^{-\lambda t} \frac{(\lambda t)^k}{k!}.$$

Thus

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Examples

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(b) The number of α-particles discharged from some radioactive material in a fixed period of time.

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Suppose that independent trials, each results in a success with probability $p \in (0, 1)$ and a failure with probability $1 - p$, are performed until a success occurs. Let X be the number of trials need, then

$$P(X = n) = (1 - p)^{n-1} p, \quad n = 1, 2, \ldots.$$

Such a random variable is called a geometric random variable with parameter p.

If X is a geometric random variable with parameter p, then for any $k \geq 1$,

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Example 1

Cards are randomly selected from an ordinary deck, one at a time, until a spade is obtained. If we assume that each card is returned to the deck before the next one is selected, find the probability that (a) exactly 10 cards are needed; (b) at least 10 cards are needed.

Solution

The number of cards needed is a geometric random variable with parameter $\frac{1}{4}$. Thus (a) $(\frac{3}{4})^9(\frac{1}{4})$; (b) $(\frac{3}{4})^9$.

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X is a geometric random variable with parameter p, then

$$E[X] = \frac{1}{p}, \quad \mathrm{Var}(X) = \frac{1-p}{p^2}.$$

Let $q = 1 - p$. Then

$$\begin{aligned} E[X] &= \sum_{n=1}^{\infty} nq^{n-1}p = p\sum_{n=0}^{\infty} \frac{d}{dq}(q^n) \\ &= p\frac{d}{dq}\left(\sum_{n=0}^{\infty} q^n\right) = p\frac{d}{dq}\left(\frac{1}{1-q}\right) \\ &= \frac{p}{(1-q)^2} = \frac{1}{p}. \end{aligned}$$

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