Math 461 Fall 2021

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University of Illinois at Urbana-Champaign

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Outline

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Definition

If $P(F) > 0$, we define

$$P(E|F) = \frac{P(E \cap F)}{P(F)}.$$

If $P(F) = 0$, $P(E|F)$ is undefined.

Using the definition of conditional probability, one can easily check

$$P(E \cap F) = P(F)P(E|F).$$

More generally, we have

$$P(\cap_{i=1}^{n} E_i) = P(E_1)P(E_2|E_1) \cdots P(E_n| \cap_{i=1}^{n-1} E_i).$$

These formulas are very useful in finding the probability of intersections.

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These formulas are very useful in finding the probability of intersections.

Example 3

Suppose that a box contains 8 red balls and 4 white balls. We randomly draw two balls from the box without replacement. Find the probability that (a) both balls are red; (b) the second ball is red.

(a) Using the obvious notation,

$$P(R_1 \cap R_2) = P(R_1)P(R_2|R_1) = \frac{8}{12}\frac{7}{11}.$$

$$P(R_2) = P(R_1 \cap R_2) + P(W_1 \cap R_2) = \frac{8}{12}\frac{7}{11} + \frac{4}{12}\frac{8}{11}.$$

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Example 3

Suppose that in the previous example. 3 balls are randomly selected from the box without replacement. Find the probability that all 3 are red.

$$P(R_1 \cap R_2 \cap R_3) = P(R_1)P(R_2|R_1)P(R_3|R_1 \cap R_2)$$
$$= \frac{8}{12}\frac{7}{11}\frac{6}{10}.$$

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Example 4

An ordinary deck of 52 cards is randomly divided into 4 distinct piles of 13 each. Find the probability that each pile has exactly 1 ace.

Solution. From an example in Section 2.5, we know that the answer is

$$\frac{4!\binom{48}{12,12,12,12}}{\binom{52}{13,13,13,13}}$$

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Solution by conditional probability. For $i = 1, 2, 3, 4$, let E_i be the event that the i-th pile has exactly 1 ace. Then

$$P(E_1) = \frac{4\binom{48}{12}}{\binom{52}{13}}, \quad P(E_2|E_1) = \frac{3\binom{36}{12}}{\binom{39}{13}},$$

$$P(E_3|E_1 \cap E_2) = \frac{2\binom{24}{12}}{\binom{26}{13}}, \quad P(E_4|E_1 \cap E_2 \cap E_3) = \frac{\binom{12}{12}}{\binom{13}{13}} = 1.$$

So the answer is

$$\frac{4\binom{48}{12}}{\binom{52}{13}} \frac{3\binom{36}{12}}{\binom{39}{13}} \frac{2\binom{24}{12}}{\binom{26}{13}} \frac{\binom{12}{12}}{\binom{13}{13}}.$$

The 2 answers are the same. See the book for yet another solution via conditional probability.

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Example 1

A certain blood test is 95% effective in detecting a certain disease when it is in fact present. However, the test also yields a "false positive" result for 1% of the healthy people tested. If 0.5% of the population has the disease, what is the probability that a person has the disease given that the person's test result is positive?

Solution. Let E be the event that the person has the disease, and F the event that the person's test result is positive. We are looking for $P(E|F)$, which is equal to

$$\frac{P(E \cap F)}{P(F)}.$$

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Solution. Let E be the event that the person has the disease, and F the event that the person's test result is positive. We are looking for $P(E|F)$, which is equal to

$$\frac{P(E \cap F)}{P(F)}.$$

We are given

$$P(E) = 0.005, \quad P(E^c) = 0.995$$

and

$$P(F|E) = .95 \quad P(F|E^c) = .01.$$

Thus

$$P(E \cap F) = P(E)P(F|E) = (0.005) \cdot (0.95)$$

and

$$\begin{aligned} P(F) = P(E \cap F) + P(E^c \cap F) &= P(E)P(F|E) + P(E^c)P(F|E^c) \\ &= (0.005) \cdot (0.95) + (0.995) \cdot (0.01). \end{aligned}$$

The answer is

$$\frac{(0.005) \cdot (0.95)}{(0.005) \cdot (0.95) + (0.995) \cdot (0.01)} \approx 0.323.$$

The example above is a special case of the following general situation. Suppose $A_1, A_2, \ldots, A_n$ are n disjoint events with their union being the whole sample space and with $P(A_i) > 0$ for each $i = 1, \ldots, n$. Let B be an event with $P(B) > 0$. Suppose that $P(A_i), P(B|A_i), i = 1, \ldots, n$ are given. Find $P(A_i|B)$.

$$B = B \cap \left(\cup_{j=1}^{n} A_j\right) = \cup_{j=1}^{n}(B \cap A_j).$$

So

$$P(B) = \sum_{j=1}^{n} P(A_j)P(B|A_j).$$

Thus

$$P(A_i|B) = \frac{P(A_i \cap B)}{P(B)} = \frac{P(A_i)P(B|A_i)}{\sum_{j=1}^{n} P(A_j)P(B|A_j)}.$$

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The formula above is known as the Bayes' formula. You do not need to memorize this formula. It is much easier to remember the short derivation of it.

Example 2

In answering a certain multiple choice question with 5 possible answers, a student either knows the answer or guesses. Assume that a student knows the answer with probability 0.8. Assume that, when not knowing the answer, the student guesses the 5 answers with equal probability. Find the probability that the student knows the answer given that the student answered it correctly.

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Solution. Let K be the event that the student knows the answer, and C the event that the student answered it correctly. Then

$$P(K) = 0.8, \quad P(K^c) = 0.2$$

and

$$P(C|K) = 1 \quad P(C|K^c) = 0.2.$$

So

$$P(K|C) = \frac{P(K \cap C)}{P(C)} = \frac{P(K)P(C|K)}{P(K)P(C|K) + P(K^c)P(C|K^c)}$$
$$= \frac{(0.8) \cdot 1}{(0.8) \cdot 1 + (0.2) \cdot (0.2)}.$$

Example 3

Suppose that there are 3 chests of drawers and each chest has 2 drawers. The first chest has a gold coin in each drawer; the second chest has a gold in one drawer and a silver coin in the other; the third chest has a silver coin in each drawer. A chest is chosen at random and a drawer is randomly opened. If the drawer has a gold coin, what is the probability that the other drawer also has a glod coin?

Solution. For $i = 1, 2, 3$, let E_i be the event that the i-th chest is chosen, and let G be the event that the drawer opened has a gold coin. We are looking for $P(E_1|G)$.

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$$P(E_1) = P(E_2) = P(E_3) = \frac{1}{3};$$

and

$$P(G|E_1) = 1, \quad P(G|E_2) = \frac{1}{2}, \quad P(G|E_3) = 0.$$

So

$$\begin{aligned} P(E_1|G) &= \frac{P(E_1 \cap G)}{P(E_1 \cap G) + P(E_2 \cap G) + P(E_3 \cap G)} \\ &= \frac{P(E_1)P(G|E_1)}{P(E_1)P(G|E_1) + P(E_2)P(G|E_2) + P(E_3)P(G|E_3)} \\ &= \frac{\frac{1}{3}}{\frac{1}{3} + \frac{1}{3}\frac{1}{2}} = \frac{2}{3}. \end{aligned}$$

A plane is missing, and it is presumed that it is equally likely to have gone down in any of 3 possible regions. Let $1-\beta_i$ be the probability that the plane will be found upon a search of the region when the plane is, in fact, in that region, $i =, 2, 3$. Find the probability that the plane is in the i-th region given that a search of region 1 did not locate the plane.

Solution. For $i = 1, 2, 3$, let E_i be the event that the plane is the i-th region. Let F be the event that a search of region 1 did not locate the plane. We are looking for $P(E_1|F)$, $P(E_2|F)$ and $P(E_3|F)$.

$$P(E_1) = P(E_2) = P(E_3) = \frac{1}{3},$$

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So

$$P(E_1|F) = \frac{P(E_1 \cap F)}{P(E_1 \cap F) + P(E_2 \cap F) + P(E_3 \cap F)}$$
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Similarly,

$$P(E_2|F) = P(E_3|F) = \frac{1}{\beta_1 + 2}.$$

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