

Math 461 Fall 2021

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University of Illinois at Urbana-Champaign

August 27, 2021

Outline

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- 1 **General Info**
- 2 2.2 Sample Spaces
- 3 2.3 Axioms of Probability
- 4 2.4 Some Simple Propositions

I have setup a Moodle page. I have put the links to the slides of the lecture there. Please take a look to see if things work as intended.

I have also setup HW1 there, you will need to submit your homework from that page if you want to submit your HW electronically. The deadline for submitting HW1 is next Friday, 09/03, before the end of our lecture.

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There are lots of phenomena in nature whose outcome cannot be predicted with certainty in advance, but the set of all the possible outcomes is known. For instance, when you toss a coin, you do not know whether “Heads” or “Tails” will appear, but you do know the outcome will be either “Heads” or “Tails”. These are what we call random phenomena or random experiments. Probability theory is concerned with such random experiments.

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Tossing a coin. $S = \{H, T\}$. H stands for “Heads”, and T stands for “Tails”.

Tossing a (6-sided) die. $S = \{1, 2, 3, 4, 5, 6\}$.

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Measuring the lifetime of a light-bulb. $S = [0, \infty)$.

Any subset E of the sample space S is known as an *event*. Some examples are

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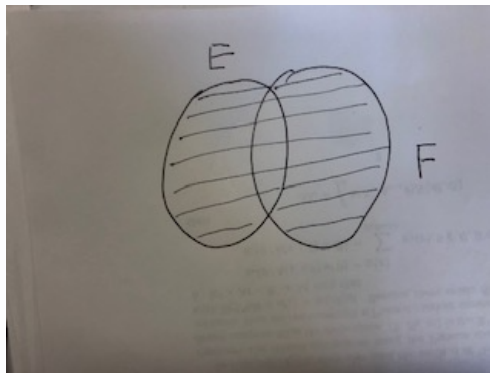
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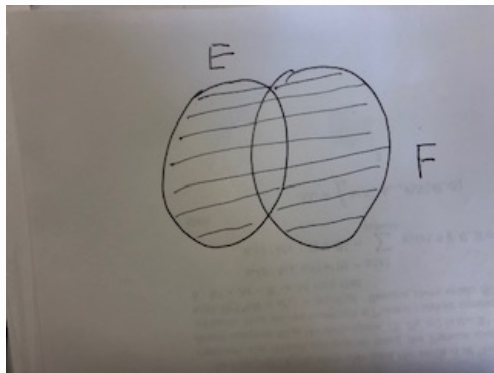
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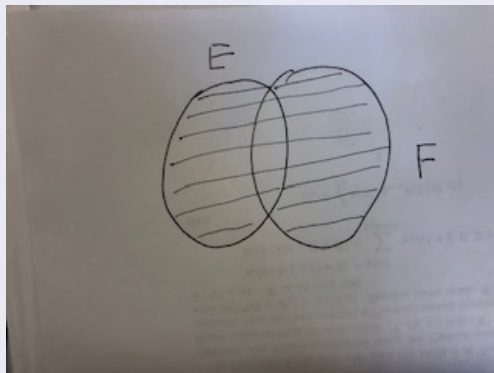
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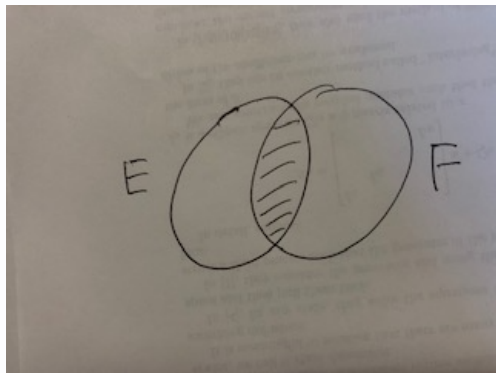


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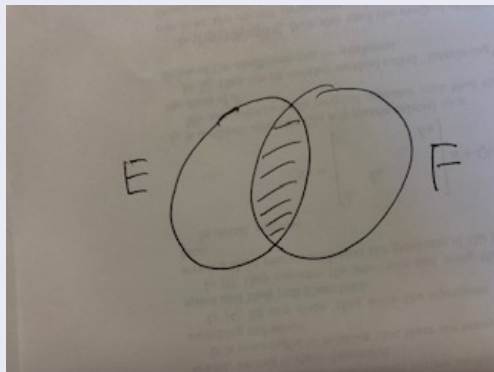
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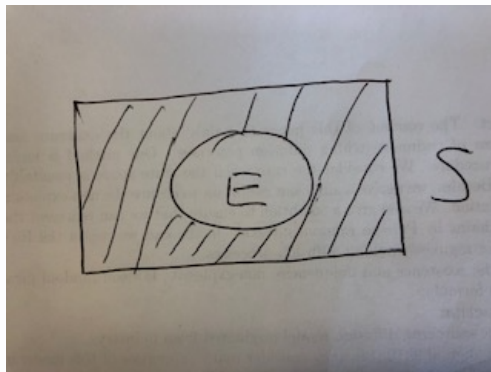
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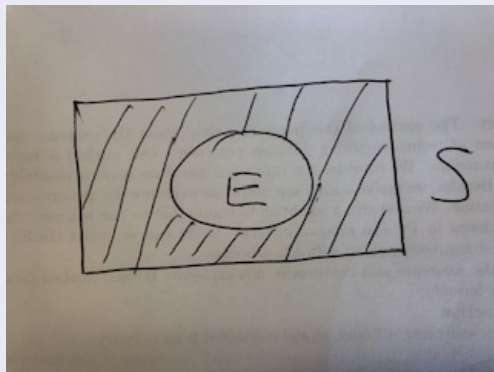
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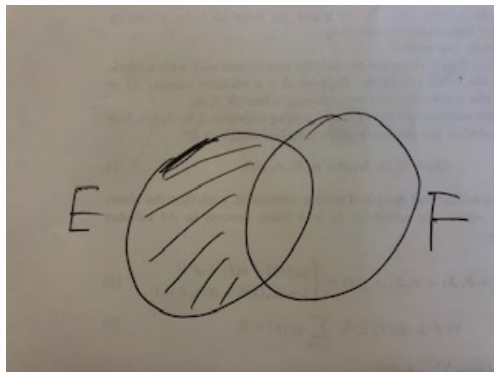
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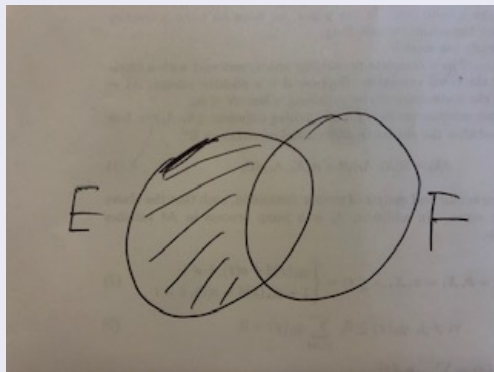
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$E \subset F$ if every element of E is an element of F .

If $E \cap F = \emptyset$, then we say that E and F are disjoint, or mutually exclusive.

Similarly, we can define the union and intersection of more than 2 events

$$\bigcup_{i=1}^n E_i, \quad \bigcup_{i=1}^{\infty} E_i$$

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Properties of set theoretical operations

Commutativity: $E \cup F = F \cup E$ and $E \cap F = F \cap E$;

Associativity: $(E \cup F) \cup G = E \cup (F \cup G)$ and
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Distributivity: $(E \cup F) \cap G = (E \cap G) \cup (F \cap G)$ and
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De Morgan's law

$$(\cup_{i=1}^n E_i)^c = \cap_{i=1}^n E_i^c \quad (\cup_{i=1}^{\infty} E_i)^c = \cap_{i=1}^{\infty} E_i^c$$

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Consider a random experiment whose sample space is S . A real-valued function P on the space of all events of the experiment is called a probability (measure) if

- (1) for all event E , $0 \leq P(E) \leq 1$;
- (2) $P(S) = 1$;
- (3) for any sequence E_1, E_2, \dots of mutually disjoint events,

$$P(\cup_{i=1}^{\infty} E_i) = \sum_{i=1}^{\infty} P(E_i).$$

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Tossing a coin for which Heads is twice likely as Tails. $P(H) = \frac{2}{3}$,
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Measuring the lifetime of a light-bulb. $P(A) = \int_A e^{-t} dt$ for any subset A of \mathbb{R}_+ .

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Properties of Probability Measures

Suppose that P is a probability measure. Then

- (1) $P(\emptyset) = 0$;
- (2) if E_1, \dots, E_n are disjoint, then

$$P(\cup_{i=1}^n E_i) = \sum_{i=1}^n P(E_i);$$

- (3) if $E \subset F$, the $P(E) \leq P(F)$;
- (4) $P(E^c) = 1 - P(E)$;
- (5) $P(\cup_{i=1}^n E_i) = 1 - P(\cap_{i=1}^n E_i^c)$;
- (6) $P(E \cup F) = P(E) + P(F) - P(E \cap F)$.

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Proof

(1) Take $E_1 = E_2 = \dots = \emptyset$, then

$$P(\emptyset) = P(\cup_{i=1}^{\infty} E_i) = P(\emptyset) + P(\emptyset) + \dots,$$

so $P(\emptyset) = 0$.

(2) Take $E_{n+1} = E_{n+2} = \dots = \emptyset$, then E_1, E_2, \dots is a sequence of disjoint events, thus by countable additivity,

$$P(\cup_{i=1}^n E_i) = P(\cup_{i=1}^{\infty} E_i) = \sum_{i=1}^{\infty} P(E_i) = \sum_{i=1}^n P(E_i).$$

(3) $P(F) = P(E) + P(F \setminus E) \geq P(E)$.

(4) $1 = P(E \cup E^c) = P(E) + P(E^c)$.

Proof (cont)

(5) Follows immediately from (4),

(6) Let $I = E \setminus F$, $II = F \setminus E$ and $III = E \cap F$. Then

$P(E \cup F) = P(I \cup II \cup III) = P(I) + P(II) + P(III)$ and

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Thus $P(E \cup F) = P(E) + P(F) - P(E \cap F)$.

Example

A fair die is tossed 100 times. Find the probability that there is at least one 5.

The complement of "at least one 5" is "there is no 5". So the answer is

$$1 - \left(\frac{5}{6}\right)^{100}.$$

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A generalization of (6) to the case of the union of n events is the following inclusion-exclusion formula, which can be proved by induction.

Inclusion-exclusion formula

If E_1, E_2, \dots, E_n are events, then

$$\begin{aligned}
 P(\cup_{i=1}^n E_i) &= \sum_{i=1}^n P(E_i) - \sum_{i_1 < i_2} P(E_{i_1} \cap E_{i_2}) + \sum_{i_1 < i_2 < i_3} P(E_{i_1} \cap E_{i_2} \cap E_{i_3}) \\
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