# Math 461 Fall 2021

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August 25, 2021

# Outline

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Some homework assignments are posted in the course page in the my homepage. The first set is due next Friday, 09/03.

The slides of the first lecture is also posted in the course page.

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## Example 1

Consider a set of $n$ antennas, of which $m$ are defective and $n-m$ are functional. Assume $m \leq n-m+1$. Assume also that all of the defective ones are indistinguishable, and all the functional ones are indistinguishable. How many linear orderings are there in which no 2 defectives ones are consecutive?

Imagine that the $n-m$ functional antennas are lined up. Now if no 2 defectives ones are to be consecutive, then the spaces between the functional antennas must contain at most 1 defective antenna. That is in the $n-m+1$ possible positions, we must select $m$ of which to put n the defective antennas. So the answer is

$$\binom{n-m+1}{m}.$$

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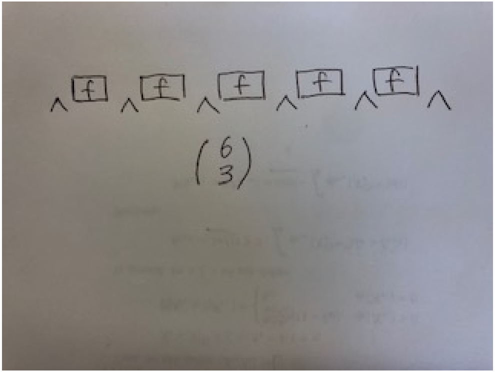
Here is an illustration with $n = 8$ and $m = 3$.



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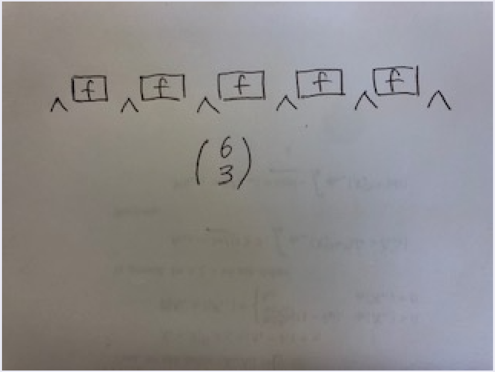

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$$\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r}.$$

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The values $\binom{n}{r}$ are often called the binomial coefficients. This is because of

**Binomial Theorem**

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}.$$

As a consequence of the binomial theorem, we have

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n.$$

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You can prove the the binomial theorem using induction. Here I give a combinatorial proof.

**Proof of the Binomial Theorem**

Consider the product:

$$(x_1 + y_1)(x_2 + y_2) \cdots (x_n + y_n).$$

Its expansion is the sum of $2^n$ terms, each term being the product of $n$ factors. Furthermore, each of the $2^n$ terms in the sum will contain as a factor either $x_i$ or $y_i$ for each $i = 1, \ldots, n$. How many of the the $2^n$ terms have as factors $k$ of the $x_i$'s and $(n-k)$ of the $y_i$'s? Answer: $\binom{n}{k}$. Thus, letting $x_i = x, y_i = y$, $i = 1, \ldots, n$, we get

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A set of $n$ distinct items is to be divided into $r$ distinct groups of sizes $n_1, \ldots, n_r$, where $n_i \geq 0, i = 1, \ldots r$ and $\sum_{i=1}^{r} n_i = n$. How many different divisions are there?

Answer:

$$\binom{n}{n_1}\binom{n-n_1}{n_2}\cdots\binom{n-n_1-\cdots-n_{r-1}}{n_r} = \frac{n!}{n_1!n_2!\cdots n_r!}.$$

Notation:

$$\binom{n}{n_1, n_2, \cdots, n_r} = \frac{n!}{n_1!n_2!\cdots n_r!}.$$

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The quantities above are often called the multinomial coefficients because of the

**Multinomia Theorem**

$$(x_1 + \cdots + x_r)^n = \sum_{(n_1,\cdots,n_r):n_i \geq 0, n_1+\cdots+n_r=n} \binom{n}{n_1, n_2, \cdots, n_r} x_1^{n_1} \cdots x_r^{n_r}.$$

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One can give a combinatorial proof of this, similar to the case of the binomial theorem.

Question: How many terms are there on the right hand side of the multinomial theorem? We will come back to these a little later.

**Example 2**

Expanding $(a + b + c + d)^{10}$ will take quite some time. What is the coefficient of $a^2 b^3 c^4 d$?

$$\binom{10}{2, 3, 4, 1}.$$

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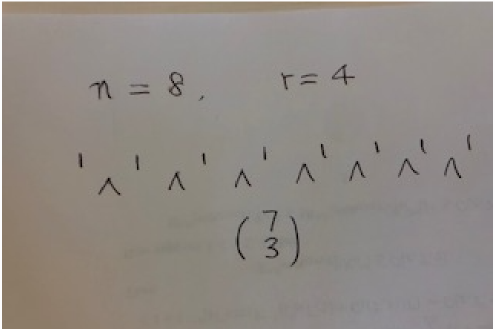
Suppose that we have $n$ indistinguishable balls. How many ways can we divide them into $r$ distinct non-empty groups (distribute them into $r$ distinct boxes so that no box is empty)?

Line up the balls and choose the $r-1$ division lines:

$$\binom{n-1}{r-1}.$$



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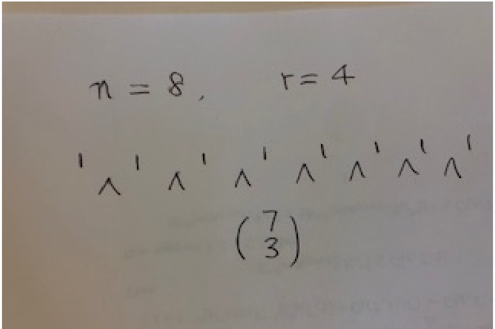

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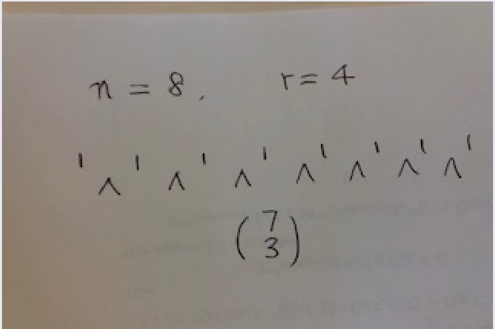

Another way of stating the result above is: There are $\binom{n-1}{r-1}$ integer-valued vectors $(x_1, \ldots, x_r)$ satisfying

$$x_1 + \cdots + x_r = n, \text{ and } x_i > 0, i = 1, \ldots, r.$$

Now let's change things a little bit. How many integer-valued vectors $(x_1, \ldots, x_r)$ are there such that

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Answer:

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The number of integer-valued vectors $(x_1, \ldots, x_r)$ satisfying (1) is the same as the number of integer-valued vectors $(y_1, \ldots, y_r)$ satisfying

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$(x_1, \ldots, x_r)$ satisfies (1) if and only if $(x_1 + 1, \ldots, x_r + 1)$ satisfies (2).

There are $\binom{n+r-1}{r-1}$ terms in the expansion of $(x_1 + \cdots + x_r)^n$. In particular, there are $\binom{13}{3}$ terms in the expansion of $(a + b + c + d)^{10}$.

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