Math 461 Fall 2021

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Outline
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1. General Info
2. 6.2 Independent random variable
3. 6.3 Sums of independent random variables.
HW7 is due Friday, 10/22, before the end of class time. Please submit your HW7 via the course Moodle page.

Solution to HW6 is on my homepage now.
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1. General Info

2. 6.2 Independent random variable

3. 6.3 Sums of independent random variables.
Two random variables $X$ and $Y$ are said to be independent if for any two subsets $A$ and $B$ of $\mathbb{R}$,

$$P(X \in A, Y \in B) = P(X \in A)P(Y \in B).$$

It can be shown that $X$ and $Y$ are independent if and only if

$$F(x, y) = F_X(x)F_Y(y), \quad (x, y) \in \mathbb{R}^2.$$
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It can be shown if $X$ and $Y$ are discrete random variables with joint mass function $p(\cdot, \cdot)$, then $X$ and $Y$ are independent if and only if
\[ p(x, y) = p_X(x)p_Y(y), \quad (x, y) \in \mathbb{R}^2. \]

It can be shown if $X$ and $Y$ are jointly absolutely continuous with joint density $f(\cdot, \cdot)$, then $X$ and $Y$ are independent if and only if
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$$f(x, y) = f_X(x)f_Y(y), \quad (x, y) \in \mathbb{R}^2.$$
Example 2

Suppose that the number of people entering a certain post office on a given day is a Poisson random variable with parameter $\lambda > 0$. Assume that each person entering the post office is male with probability $p$ and female with probability $1 - p$, independent of all others. Show that the number of males and the number of females entering the post office on a given day are independent Poisson random variables with parameters $\lambda p$ and $\lambda (1 - p)$ respectively.

Let $X$ and $Y$ be the number of males and the number of females entering the post office on a given day respectively. $X$ and $Y$ are non-negative integer-valued random variables.
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Let $X$ and $Y$ be the number of males and the number of females entering the post office on a given day respectively. $X$ and $Y$ are non-negative integer-valued random variables.
For any non-negative integers $i$ and $j$,

$$P(X = i, Y = j) = P(X = i, Y = j | X + Y = i + j)P(X + Y = i + j)$$

$$= \binom{i + j}{i} p^i (1 - p)^j e^{-\lambda} \frac{\lambda^{i+j}}{(i+j)!}$$

$$= e^{-\lambda p} \frac{(\lambda p)^i}{i!} e^{-\lambda (1-p)} \frac{\lambda (1-p)^j}{j!}.$$ 

Hence

$$P(X = i) = e^{-\lambda p} \frac{(\lambda p)^i}{i!} \sum_{j=0}^{\infty} e^{-\lambda (1-p)} \frac{\lambda (1-p)^j}{j!} = e^{-\lambda p} \frac{(\lambda p)^i}{i!}.$$ 

Similarly

$$P(Y = j) = e^{-\lambda (1-p)} \frac{(\lambda (1-p))^j}{j!}.$$ 

Therefore $X$ and $Y$ are independent Poisson random variables with parameters $\lambda p$ and $\lambda (1 - p)$ respectively.
Example 3

A man and a woman decide to meet at a certain location. If each of them independently arrives at a time uniformly distributed between noon and 1 pm. Find the probability that the first to arrive needs to wait no more than 10 minutes.
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Area of shaded region

$$= (60)^2 - (50)^2$$
The answer is
\[
\frac{60^2 - 50^2}{60^2} = \frac{11}{36}.
\]

Proposition

(i) Suppose that \(X\) and \(Y\) are discrete with joint mass function \(p(\cdot, \cdot)\). Then \(X\) and \(Y\) are independent if and only if
\[
p(x, y) = g(x)h(y), \quad (x, y) \in \mathbb{R}^2
\]
for some functions \(g\) and \(h\) on \(\mathbb{R}\).

(ii) Suppose that \(X\) and \(Y\) are jointly abs. cont. with joint density \(f(\cdot, \cdot)\). Then \(X\) and \(Y\) are independent if and only if
\[
f(x, y) = g(x)h(y), \quad (x, y) \in \mathbb{R}^2,
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for some functions \(g\) and \(h\) on \(\mathbb{R}\).
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**Proposition**

(i) Suppose that $X$ and $Y$ are discrete with joint mass function $p(\cdot, \cdot)$. Then $X$ and $Y$ are independent if and only if

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(ii) Suppose that $X$ and $Y$ are jointly abs. cont. with joint density $f(\cdot, \cdot)$. Then $X$ and $Y$ are independent if and only if

\[ f(x, y) = g(x)h(y), \quad (x, y) \in \mathbb{R}^2, \]

for some functions $g$ and $h$ on $\mathbb{R}$. 
Example 4

The joint density of $X$ and $Y$ is

$$f(x, y) = \begin{cases} 10e^{-(2x+5y)}, & x > 0, y > 0 \\ 0, & \text{otherwise.} \end{cases}$$

If

$$g(x) = \begin{cases} 10e^{-2x}, & x > 0, \\ 0, & \text{otherwise,} \end{cases} \quad h(y) = \begin{cases} e^{-5y}, & y > 0, \\ 0, & \text{otherwise.} \end{cases}$$

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Example 5

The joint density of $X$ and $Y$ is

$$f(x, y) = \begin{cases} 24xy, & x \in (0, 1), y \in (0, 1), x + y \in (0, 1) \\ 0, & \text{otherwise.} \end{cases}$$
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$$f(x, y) = \begin{cases} 24xy, & x \in (0, 1), y \in (0, 1), x + y \in (0, 1) \\ 0, & \text{otherwise.} \end{cases}$$
Both $X$ and $Y$ take values in $(0, 1)$. For $x \in (0, 1)$,

$$f_X(x) = \int_0^{1-x} 24xy dy = 12x(1 - x)^2.$$  

Similarly, for $y \in (0, 1)$,

$$f_Y(y) = 12y(1 - y)^2.$$  

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The concept of independent random variables can be extended to more than 2 random variables.
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The concept of independent random variables can be extended to more than 2 random variables.
$n$ random variables $X_1, \ldots, X_n$ are said to be independent if for any subsets $A_1, \ldots, A_n$ of $\mathbb{R}$,

$$P(X_1 \in A_1, \ldots X_n \in A_n) = \prod_{i=1}^{n} P(X_i \in A_i).$$

It can be shown that $n$ random variables $X_1, \ldots, X_n$ with joint distribution function $F(\cdot, \ldots, \cdot)$ are independent if and only if

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Example 6

Suppose that $X_1, \ldots, X_n$ are independent absolutely continuous random random variables with a common density $f$. Define

$$U = \min\{X_1, \ldots, X_n\}, \quad V = \max\{X_1, \ldots, X_n\}.$$ 

Find the densities of $U$ and $V$ respectively.

Let’s deal with $V$ first. Let $F$ be the common distribution. For any $v \in \mathbb{R}$,

$$P(V \leq v) = P(X_1 \leq v, \ldots, X_n \leq v) = (F(v))^n.$$ 

Thus the density of $V$ is $f_V(v) = n(F(v))^{n-1}f(v)$. 
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Thus the density of $V$ is $f_V(v) = n(F(v))^{n-1}f(v)$. 
Now let’s deal with $U$. For any $u \in \mathbb{R}$,

\[
P(U \leq u) = 1 - P(U > u) = 1 - P(X_1 > u, \ldots, X_n > u) = 1 - (1 - F(u))^n.
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Thus the density of $U$ is

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We can also find the joint density of $U$ and $V$. I will come back to this later in this chapter.
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For any $z \in \mathbb{R}$,

$$F_z(z) = P(X + Y \leq z)$$

$$= \int_{-\infty}^{\infty} \left( \int_{-\infty}^{z-x} f_X(x) f_Y(y) dy \right) dx$$

$$= \int_{-\infty}^{\infty} \left( \int_{-\infty}^{z} f_X(x) f_Y(v - x) dv \right) dx, \quad (y = v - x)$$

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