HW6 is due Friday, 10/15, before the end of the class. You can either submit your HW6 via the course Moodle page, or submit a hard copy.

Solution to Test 1 is on my homepage.
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Outline

1. General Info

2. 5.6 Other Absolutely Continuous Random Variables

3. 5.7 The distribution of a function of a random variable
For any $\alpha > 0$, the Gamma function is defined as

$$\Gamma(\alpha) = \int_0^\infty y^{\alpha-1} e^{-y} dy.$$ 

By using a simple change of variables, one can check that, for any $\alpha > 0$ and $\lambda > 0$, the function

$$f(x) = \begin{cases} \frac{1}{\Gamma(\alpha)} (\lambda x)^{\alpha-1} e^{-\lambda x}, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

is a probability density. It is called a Gamma density with parameters $(\alpha, \lambda)$. 
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is a probability density. It is called a Gamma density with parameters \((\alpha, \lambda)\).
A random variable $X$ is called a Gamma random variable with parameters $(\alpha, \lambda)$ if it is an absolutely continuous random variable whose density is a Gamma density with parameters $(\alpha, \lambda)$.

If $X$ is a Gamma random variable with parameters $(\alpha, \lambda)$, then

$$E[X] = \frac{\alpha}{\lambda}, \quad \text{Var}(X) = \frac{\alpha}{\lambda^2}.$$
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If $X$ is a Gamma random variable with parameters $(\alpha, \lambda)$, then

$$E[X] = \frac{\alpha}{\lambda}, \quad \text{Var}(X) = \frac{\alpha}{\lambda^2}.$$
\[ E[X] = \frac{1}{\Gamma(\alpha)} \int_0^\infty x(\lambda x)^{\alpha-1} \lambda e^{-\lambda x} \, dx \]
\[ = \frac{1}{\lambda \Gamma(\alpha)} \int_0^\infty (\lambda x)^\alpha \lambda e^{-\lambda x} \, dx \]
\[ = \frac{\Gamma(\alpha + 1)}{\lambda \Gamma(\alpha)} = \frac{\alpha}{\lambda}. \]

\[ E[X^2] = \frac{1}{\Gamma(\alpha)} \int_0^\infty x^2(\lambda x)^{\alpha-1} \lambda e^{-\lambda x} \, dx \]
\[ = \frac{1}{\lambda^2 \Gamma(\alpha)} \int_0^\infty (\lambda x)^{\alpha+1} \lambda e^{-\lambda x} \, dx \]
\[ = \frac{\Gamma(\alpha + 2)}{\lambda^2 \Gamma(\alpha)} = \frac{\alpha(\alpha + 1)}{\lambda^2}. \]
By using the same argument, we can find that, for any $n = 1, 2, \ldots$,

$$E[X^n] = \frac{\alpha(\alpha + 1) \cdots (\alpha + n - 1)}{\lambda^n}.$$

Example 1

Suppose that $X$ is an absolutely continuous random variable with density $f$. Find the density of $Y = X^2$.

$Y$ is a positive random variable. For any $y > 0$,

$$P(Y \leq y) = P(X^2 \leq y) = P(-\sqrt{y} \leq X \leq \sqrt{y})$$

$$= \int_{-\sqrt{y}}^{\sqrt{y}} f(x) \, dx.$$
By using the same argument, we can find that, for any $n = 1, 2, \ldots$,

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Differentiate with respect to $y$ and using the second fundamental theorem of calculus, we find that the density of $Y$ is

$$f_Y(y) = \begin{cases} \frac{1}{2\sqrt{y}}(f(\sqrt{y}) + f(-\sqrt{y})), & y > 0 \\ 0, & y \leq 0. \end{cases}$$

In the particular case when $X$ is a standard normal random variable, the density of $Y$ is

$$f_Y(y) = \begin{cases} \frac{1}{\sqrt{2\pi}} y^{-1/2} e^{-\frac{y}{2}}, & y > 0 \\ 0, & y \leq 0. \end{cases}$$
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By comparing with the Gamma density with parameters \(\left(\frac{1}{2}, \frac{1}{2}\right)\)

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g(y) = \begin{cases} 
\frac{1}{\Gamma(1/2)} \left(\frac{y}{2}\right)^{1/2-1} \frac{1}{2} e^{-\frac{y}{2}}, & y > 0 \\
0, & y \leq 0, 
\end{cases}
\]

we get that, when \(X\) is a standard normal random variable, \(X^2\) is a Gamma random variable with parameters \(\left(\frac{1}{2}, \frac{1}{2}\right)\).

Furthermore, we can get

\[
\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}.
\]

Combining this with \(\Gamma(\alpha + 1) = \alpha \Gamma(\alpha)\), we get

\[
\Gamma\left(\frac{n}{2}\right) = \frac{\sqrt{\pi(n - 1)!}}{2^{n-1}\left(\frac{n-1}{2}\right)!}, \quad n \text{ odd}.
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Similarly, we get that, when $X$ is a normal random variable with parameters $(0, \sigma^2)$, $X^2$ is a Gamma random variable with parameters $(\frac{1}{2}, \frac{1}{2}\sigma^2)$.

For any $a, b > 0$,

$$B(a, b) = \int_{0}^{1} x^{a-1}(1 - x)^{b-1} dx$$

is a finite positive number.
Similarly, we get that, when $X$ is a normal random variable with parameters $(0, \sigma^2)$, $X^2$ is a Gamma random variable with parameters $(\frac{1}{2}, \frac{1}{2\sigma^2})$.

For any $a, b > 0$,

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is a finite positive number.
Thus, for any $a, b > 0$, the function

$$f(x) = \begin{cases} \frac{1}{B(a,b)} x^{a-1} (1 - x)^{b-1}, & x \in (0, 1) \\ 0, & \text{otherwise} \end{cases}$$

is a probability density. It is called a Beta density with parameters $(a, b)$.

A random variable $X$ is called a Beta random variable with parameters $(a, b)$ if it is an absolutely continuous random variable whose density is a Beta density with parameters $(a, b)$. 
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A random variable $X$ is called a Beta random variable with parameters $(a, b)$ if it is an absolutely continuous random variable whose density is a Beta density with parameters $(a, b)$. 
The Beta function \( B(a, b) \) and the Gamma function are related by

\[
B(a, b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}
\]

Combining this with the fact that \( \Gamma(\alpha + 1) = \alpha \Gamma(\alpha) \), we can get

If \( X \) is a Beta random variable with parameters \((a, b)\), then

\[
E[X] = \frac{a}{a+b}, \quad \text{Var}(X) = \frac{ab}{(a+b)^2(a+b+1)}.
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If $X$ is a Beta random variable with parameters $(a, b)$, then

$$E[X] = \frac{a}{a+b}, \quad \text{Var}(X) = \frac{ab}{(a+b)^2(a+b+1)}.$$
Suppose that $X$ is an absolutely continuous random variable with density $f$ and $Y = \phi(X)$ for some function $\phi$. Find the density of $Y$.

Although there is a theorem in the book, I recommend that you follow the following procedure:
(1) Find the distribution of $Y$;
(2) Differentiate.
Suppose that \( X \) is an absolutely continuous random variable with density \( f \) and \( Y = \phi(X) \) for some function \( \phi \). Find the density of \( Y \).

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(1) Find the distribution of \( Y \);
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Example 1

Suppose that $X$ is a uniform random variable on $(0, 1)$, and $\lambda > 0$. Find the density of $Y = -\frac{1}{\lambda} \ln(1 - X)$.

$Y$ is a non-negative random variable. For any $y > 0$,

$$P(Y \leq y) = P\left(-\frac{1}{\lambda} \ln(1 - X) \leq y\right) = P(\ln(1 - X) \geq -\lambda y)$$

$$= P(1 - X \geq e^{-\lambda y}) = P(X \leq 1 - e^{-\lambda y}) = 1 - e^{-\lambda y}.$$  

Thus the density of $Y$ is

$$f_Y(y) = \begin{cases} 
\lambda e^{-\lambda y}, & y > 0 \\
0, & y \leq 0.
\end{cases}$$

$Y$ is an exponential random variable with parameter $\lambda$. 
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Example 2

Suppose that $X$ is an exponential random variable with parameter $\lambda$ and $\beta \neq 0$. Find the density of $Y = X^{1/\beta}$.

$Y$ is a non-negative random variable. Let consider the case $\beta < 0$. For any $y > 0$,

$$P(Y \leq y) = P(X^{1/\beta} \leq y) = P(X \geq y^\beta) = e^{-\lambda y^\beta}.$$ 

Thus the density of $Y = X^{1/\beta}$ is

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Using a similar derivation, we can find that in the case \( \beta > 0 \), the density of \( Y = X^{1/\beta} \) is

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Thus, in the general case, the density of \( Y = X^{1/\beta} \) is

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