8th Homework Set — Solutions
Chapter 6

Problem 6.11 Let $A$ be the number of people buying an ordinary set, $B$ the number of people buying a plasma set, and $C$ the number of people who are just browsing. Then $P\{A = 2, B = 1, C = 2\} = \frac{5!}{2!1!2!}0.45^2 \cdot 0.15 \cdot 0.4^2 = 0.1458$.

Problem 6.13 Let $X$ be uniform on $(-15, 15)$, and let $Y$ be uniform on $(-30, 30)$. Nobody waits longer than five minutes if $|Y - X| < 5$.

\[
P\{|Y - X| < 5\} = P\{-5 < Y - X < 5\} \]
\[
= P\{X - 5 < Y < X + 5\} \]
\[
= \int_{-15}^{15} \int_{x-5}^{x+5} \frac{1}{30 \cdot 60} dy dx \]
\[
= \frac{30 \cdot 10}{30 \cdot 60} = \frac{1}{6}.
\]

The probability that the man arrives first is $P\{X < Y\} = \frac{1}{2}$ by symmetry.

Problem 6.14 Let $X, Y$ be uniform random variables on $(0, L)$. Let $Z = |Y - X|$. We want to find $E[Z]$. First, find $F_Z(a)$, for $a \geq 0$. We have $F_Z(a) = P\{Z \leq a\} = P\{|Y - X| \leq a\} = P\{-a \leq Y - X \leq a\} = \frac{2aL - a^2}{L^2}$. Using geometric considerations. Hence, $f_Z(x) = \frac{2L - 2x}{L^2}$ if $0 \leq a \leq L$. Hence,

\[
E[Z] = \int_0^L x \cdot \frac{2L - 2x}{L^2} dx \]
\[
= \frac{2}{L^2} \left( \frac{Lx^2}{2} - \frac{x^3}{3} \right) \bigg|_0^L \]
\[
= \frac{L}{3}.
\]

Problem 6.18 Let $X$ be uniform on $(0, \frac{L}{2})$ and let $Y$ be uniform on $(\frac{L}{2}, L)$. We want
to find $P\{Y - X > \frac{L}{3}\}$.

$$
P\left\{ Y - X > \frac{L}{3} \right\} = P\left\{ Y < \frac{L}{2} + \frac{L}{3}, X < Y - \frac{L}{3} \right\} + P\left\{ Y > \frac{L}{2} + \frac{L}{3} \right\}
$$

$$
= \int_{\frac{L}{2}}^{\frac{5L}{6}} \int_{0}^{\frac{y - \frac{L}{3}}{2}} \frac{4}{L^2} dx dy + \int_{\frac{5L}{6}}^{\frac{L}{2}} \frac{2}{L} dy
$$

$$
= \frac{4}{9} + \frac{1}{3} = \frac{7}{9}.
$$

Problem 6.20 If the joint density function of $X$ and $Y$ is

$$
f(x, y) = \begin{cases} xe^{-(x+y)} & x > 0, y > 0 \\ 0 & \text{otherwise}, \end{cases}
$$

then $f(x, y) = f_X(x)f_Y(y)$, where $f_X(x) = xe^{-x}$ for $x > 0$, and $f_Y(y) = e^{-y}$ for $y > 0$ (0 otherwise), so that $X$ and $Y$ are independent.

If

$$
f(x, y) = \begin{cases} 2 & 0 < x < y, 0 < y < 1 \\ 0 & \text{otherwise}, \end{cases}
$$

then $X$ and $Y$ are not independent because the nonzero values of $f$ are located in a triangular domain.

Problem 6.21 (a) Check:

$$
\int_{0}^{1} \int_{0}^{1-y} 24xydxdy = \int_{0}^{1} 12(1-y)^2ydy = 12 \int_{0}^{1} y - 2y^2 + y^3dy = 6y^2 - 8y^3 + 3y^4 \bigg|_{0}^{1} = 6 - 8 + 3 = 1.
$$

(b) First, find $f_X(x) = \int_{0}^{1-x} 24xydy = 12x(1-x)^2$. Now, $E[X] = \int_{0}^{1} 12x(1-x)^2dx = 4x^2 - 6x^3 + \frac{12}{5}x^5 \bigg|_{0}^{1} = 4 - 6 + \frac{12}{5} = \frac{2}{5}$.

(c) $E[Y] = E[X] = \frac{2}{5}$ by symmetry.

Problem 6.22 Let $X$ and $Y$ be jointly continuous with density function

$$
f(x, y) = \begin{cases} x + y & 0 < x < 1, 0 < y < 1 \\ 0 & \text{otherwise}. \end{cases}
$$

(a) $X$ and $Y$ are not independent, since $f(x, y)$ is clearly not a product of functions of $x$ and $y$.

(b) $f_X(x) = \int_{0}^{1} x + ydy = x + \frac{y^2}{2} \bigg|_{0}^{1} = x + \frac{1}{2}$.
(c) \( P \{X + Y < 1\} = \int_0^1 \int_0^{1-y} x + y \, dx \, dy = \int_0^1 \left(1 - \frac{y^2}{2}\right) + y(1-y) \, dy = \frac{1}{2} \int_0^1 1 - y^2 \, dy = \frac{1}{2} \left(1 - \frac{1}{3}\right) = \frac{1}{3}. \)

Problem 6.23 Let \( X \) and \( Y \) be jointly distributed with density function

\[
f(x, y) = \begin{cases} 
12xy(1-x) & 0 < x < 1, 0 < y < 1 \\
0 & \text{otherwise}. 
\end{cases}
\]

First, compute \( f_X(x) = \int_0^1 12xy(1-x) \, dy = 6x(1-x) \) and \( f_Y(y) = \int_0^1 12xy(1-x) \, dy = 2y. \)

(a) Clearly, \( f(x, y) = f_X(x)f_Y(y) \), so that \( X \) and \( Y \) are independent.

(b) \( E[X] = \int_0^1 6x^2(1-x) \, dx = 2x^3 - \frac{3}{2}x^4|_0^1 = \frac{1}{2}. \)

(c) \( E[Y] = \int_0^1 2y^2 \, dy = \frac{2}{3}y^3|_0^1 = \frac{2}{3}. \)

(d) First, find \( E[X^2] = \int_0^1 6x^3(1-x) \, dx = \frac{3}{2}x^4 - \frac{6}{5}x^5|_0^1 = \frac{3}{10}. \) Now, \( \text{Var}(X) = E[X^2] - EX^2 = \frac{3}{10} - \frac{1}{4} = \frac{1}{20}. \)

(e) First, find \( E[Y^2] = \int_0^1 2y^3 \, dy = \frac{1}{2}y^4|_0^1 = \frac{1}{2}. \) Now, \( \text{Var}(X) = \frac{1}{2} - \frac{4}{9} = \frac{1}{18}. \)

Problem 6.27 Let \( X_1, X_2 \) be exponential random variables with parameter \( \lambda_1, \lambda_2. \) Let \( Z = \frac{X_1}{X_2}. \) Note that \( F_Z(a) = 0 \) if \( a \leq 0. \) Compute \( F_Z(a) \) for \( a > 0: \)

\[
F_Z(a) = P \{Z \leq a\} = P \{X_1 \leq aX_2\} = \lambda_1 \lambda_2 \int_0^\infty \int_0^{ay} e^{-\lambda_1 x - \lambda_2 y} \, dx \, dy = \frac{\lambda_1 a}{\lambda_1 a + \lambda_2},
\]

so that

\[
f_Z(a) = \frac{d}{da} F(a) = \frac{\lambda_1}{\lambda_1 a + \lambda_2} - \frac{\lambda_1^2 a}{(a\lambda_1 + \lambda_2)^2}.
\]

Finally, we have

\[
P \{X_1 < X_2\} = P \{Z < 1\} = F_Z(1) = \frac{\lambda_1}{\lambda_1 + \lambda_2}.
\]
Problem 6.29 Let $X_1, X_2$ be independent normal random variables with $\mu = 2200$ and $\sigma^2 = 230^2$, representing the gross sales over this week and next week, respectively. Then $X = X_1 + X_2$ is normal with mean 4400 and variance $2 \cdot 230^2 = 105800$.

(a) $P \{X > 5000\} = P \left\{ \frac{X - 4400}{\sqrt{105800}} > \frac{600}{\sqrt{105800}} \right\} = 1 - \Phi(1.84) = 1 - 0.9671 = 0.0329$.

(b) Let $p = P \{X_1 > 2000\} = P \left\{ \frac{X_1 - 2200}{230} > \frac{-200}{230} \right\} = 1 - \Phi \left( -\frac{20}{23} \right) = \Phi(0.87) = 0.8078$.

Let $N$ be the number of weeks (out of three) in which the sales exceed $2000$. Then $N$ is binomial with parameters $(p, 3)$, so that $P \{N \geq 2\} = p^3 + 3p^2(1 - p) = 0.9034$.

Problem 6.31 Let $X$ be the number of women who never eat breakfast, and let $Y$ be the number of men who never eat breakfast. Let $Z = X + Y$. By DeMoivre-Laplace, $X$ is approximated by a normal random variable with mean $200 \cdot 0.236 = 47.2$ and variance $47.2 \cdot 0.764 = 36.061$, and $Y$ is normal with mean $200 \cdot 0.252 = 50.4$ and variance $50.4 \cdot 0.748 = 37.699$.

Let $Z_1 = X + Y$ and $Z_2 = X - Y$. Then $Z_1$ is normal with mean 97.6 and variance $36.061 + 37.699 = 73.76$, and $Z_2$ is normal with mean $-3.2$ and variance 73.76.

(a) $P \{Z_1 \geq 110\} = P \{Z_1 > 109.5\} = P \left\{ \frac{Z_1 - 97.6}{\sqrt{73.76}} > \frac{11.9}{\sqrt{73.76}} \right\} = 1 - \Phi(1.39) = 1 - 0.9177 = 0.0823$.

(b) $P \{X \geq Y\} = P \{X - Y \geq 0\} = P \{Z_2 \geq 0\} = P \{Z_2 > -0.5\} = P \left\{ \frac{Z_2 + 0.5}{\sqrt{73.76}} > \frac{2.7}{\sqrt{73.76}} \right\} = 1 - \Phi(0.31) = 0.3783$.

Problem 6.34 Let $X_1$ be the number of accidents in the next month, $X_2$ the number of accidents in the month after that, and $X_3$ the number of accidents in the third month. It makes sense to think of $X_1, X_2,$ and $X_3$ as independent Poisson random variables with parameter $\lambda = 2.2$.

Let $X = X_1, Y = X_1 + X_2,$ and $Z = X_1 + X_2 + X_3$. Then $X, Y,$ and $Z$ are Poisson with parameter 2.2, 4.4, and 6.6, respectively.

(a) $P \{X > 2\} = 1 - e^{-2.2} \left( 1 + 2.2 + \frac{2.2^2}{2} \right) = 0.3773$.

(b) $P \{Y > 4\} = 1 - e^{-4.4} \left( 1 + 4.4 + \frac{4.4^2}{2} + \frac{4.4^3}{3!} + \frac{4.4^4}{4!} \right) = 0.4488$. 

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(c) \( P \{Z > 5\} = 1 - e^{-6.6} \left( 1 + 6.6 + \frac{6.6^2}{2} + \frac{6.6^3}{3!} + \frac{6.6^4}{4!} + \frac{6.6^5}{5!} \right) = 0.6453. \)

Problem 6.38  (a) \( P \{X = i, Y = j\} = \frac{1}{5} \) for \( i = 1, \ldots, 5 \) and \( j = 1, \ldots, i, 0 \) otherwise.

<table>
<thead>
<tr>
<th>( P {X = i, Y = j} )</th>
<th>( Y = 1 )</th>
<th>( Y = 2 )</th>
<th>( Y = 3 )</th>
<th>( Y = 4 )</th>
<th>( Y = 5 )</th>
<th>( P {X = i} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( X = 1 )</td>
<td>( \frac{1}{5} )</td>
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<td>0</td>
<td>0</td>
<td>0</td>
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<tr>
<td>( X = 2 )</td>
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<td>( \frac{1}{10} )</td>
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<td>( X = 3 )</td>
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<td>( \frac{1}{15} )</td>
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<tr>
<td>( X = 4 )</td>
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<td>( \frac{1}{20} )</td>
<td>( \frac{1}{20} )</td>
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<td>( X = 5 )</td>
<td>( \frac{1}{25} )</td>
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\( P \{Y = j\} = \begin{array}{ccccc}
300 & 300 & 300 & 100 & 25 \\
1 & 1 & 1 & 1 & 1
\end{array} \)

(b) \( P \{X = i | Y = j\} = \frac{1}{\sum_{k=1}^{i} 1/k} \)

| \( P \{X = i | Y = j\} \) | \( Y = 1 \) | \( Y = 2 \) | \( Y = 3 \) | \( Y = 4 \) | \( Y = 5 \) |
|-------------------------|--------|--------|--------|--------|--------|
| \( X = 1 \)            | \( \frac{60}{137} \) | 0      | 0      | 0      |
| \( X = 2 \)            | \( \frac{30}{137} \) | \( \frac{30}{137} \) | 0      | 0      |
| \( X = 3 \)            | \( \frac{20}{137} \) | \( \frac{20}{137} \) | \( \frac{20}{137} \) | 0      |
| \( X = 4 \)            | \( \frac{15}{137} \) | \( \frac{15}{137} \) | \( \frac{15}{137} \) | \( \frac{5}{137} \) |
| \( X = 5 \)            | \( \frac{12}{137} \) | \( \frac{12}{137} \) | \( \frac{12}{137} \) | \( \frac{9}{137} \) |

(c) No.

Problem 6.40

\[ p(i, j) = \begin{bmatrix}
\frac{1}{8} & \frac{3}{8} \\
\frac{1}{3} & \frac{2}{3}
\end{bmatrix} \]

<table>
<thead>
<tr>
<th>( p(i, j) )</th>
<th>( j = 1 )</th>
<th>( j = 2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( i = 1 )</td>
<td>( \frac{1}{8} )</td>
<td>( \frac{3}{8} )</td>
</tr>
<tr>
<td>( i = 2 )</td>
<td>( \frac{1}{3} )</td>
<td>( \frac{2}{3} )</td>
</tr>
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| \( P \{X = i | Y = j\} \) | \( j = 1 \) | \( j = 2 \) |
|--------------------------|--------|--------|
| \( i = 1 \)             | \( \frac{1}{2} \) | \( \frac{1}{2} \) |
| \( i = 2 \)             | \( \frac{1}{2} \) | \( \frac{1}{2} \) |

(a) No.

(c) \( P \{XY \leq 3\} = 1 - p(2, 2) = \frac{1}{2} \)
\( P \{X + Y > 2\} = 1 - p(1, 1) = \frac{7}{9} \)
\( P \{\frac{X}{Y} > 1\} = p(2, 1) = \frac{1}{3} \)
Problem 6.41 Let $X$ and $Y$ be jointly continuous with density function $f(x,y) = xe^{-x(y+1)}$ for $x > 0, y > 0$. Note that $f_X(x) = \int_0^\infty f(x,y)dy = e^{-x}$ for $x > 0$, and $f_Y(y) = \int_0^\infty f(x,y)dx = \frac{1}{(y+1)^2}$ for $y > 0$.

(a) $f_{X|Y}(x|y) = (y + 1)^2xe^{-x(y+1)}$ for $x > 0, y > 0$, 0 otherwise, and $f_{Y|X}(y|x) = xe^{-xy}$ for $x > 0, y > 0$.

(b) Let $Z = XY$. Then for $a > 0$,

$$F_Z(a) = P\{XY < a\} = \int_0^\infty \int_0^{\frac{a}{x}} xe^{-x(y+1)}dydx = 1 - e^{-a}.$$

Hence, $f_Z(a) = \frac{d}{da}F_Z(a) = e^{-a}$ for $a > 0$, 0 otherwise.

Problem 6.42 Let $X$ and $Y$ be jointly continuous with density function

$$f(x,y) = c(x^2 - y^2)e^{-x}$$

for $0 \leq x < \infty, -x \leq y \leq x$. For $x > 0$, we have

$$f_X(x) = \int_{-x}^x c(x^2 - y^2)e^{-x}dy = \frac{4c}{3}x^3e^{-x}.$$

Hence, $f_{Y|X}(y|x) = \frac{3}{4}x^2 - \frac{y^2}{x^3}$ for $-x < y < x$, 0 otherwise. We conclude that

$$F_{Y|X}(y|x) = \begin{cases} 
0 & y \leq -x \\
\frac{3}{4} \int_{-x}^y \frac{x^2-y^2}{x^3}dy = \frac{1}{4} \left(\frac{y(3x^2-y^2)}{x^3} + 2\right) & -x < y < x \\
1 & x \leq y.
\end{cases}$$