Second Homework Set — Solutions
Chapter 2

Problem 17 There are \(64 \cdot 63 \cdot 62 \cdot 61 \cdot 60 \cdot 59 \cdot 58 \cdot 57\) ways of arranging 8 castles on a chess board. Of these, there are \(64 \cdot 49 \cdot 36 \cdot 25 \cdot 16 \cdot 9 \cdot 4 \cdot 1 = \prod_{i=1}^{8} i^2\) in which none of the rooks can capture any of the others. So the answer is

\[
\prod_{i=1}^{8} i^2 = 64 \cdot 63 \cdot 62 \cdot 61 \cdot 60 \cdot 59 \cdot 58 \cdot 57.
\]

Problem 18 \(2 \cdot 4 \cdot 16 \over 52 \cdot 51\).

Problem 20 Let \(A\) be the event that you are dealt a blackjack, and let \(B\) be the event that the dealer is dealt a blackjack.

Then

\[
P(A) = P(B) = \frac{2 \cdot 4 \cdot 16}{52 \cdot 51},
\]

\[
P(AB) = \frac{4 \cdot 4 \cdot 16 \cdot 3 \cdot 15}{52 \cdot 51 \cdot 50 \cdot 49},
\]

\[
P(A \cup B) = P(A) + P(B) - P(AB) = 0.0948.
\]

Hence, then probability that neither you nor the dealer is dealt a blackjack is \(1 - P(A \cup B) = 0.9052\).

Problem 21 (a) \(P(1) = \frac{4}{20} = \frac{1}{5}, \ P(2) = \frac{8}{20} = \frac{2}{5}, \ P(3) = \frac{5}{20} = \frac{1}{4}, \ P(4) = \frac{2}{20} = \frac{1}{10},\) and \(P(5) = \frac{1}{20}\).

(b) There are 48 children altogether, so that \(P(1) = \frac{4}{48} = \frac{1}{12}, \ P(2) = \frac{28}{48} = \frac{7}{12}, \ P(3) = \frac{35}{48} = \frac{5}{6}, \ P(4) = \frac{42}{48} = \frac{7}{8},\) and \(P(5) = \frac{3}{48}\).

Problem 25 Let \(E_n\) be the event that a sum of 5 occurs on the \(n\)th roll, and no sum of 5 or 7 occurs on the first \(n-1\) rolls. There are 36 outcomes of a single roll, and four of them give a sum of 5, while 6 of them give a sum of 7. Hence,

\[
P(E_n) = \left(\frac{26}{36}\right)^{n-1} \cdot \frac{4}{36} = \left(\frac{13}{18}\right)^{n-1} \cdot \frac{1}{9}.
\]
A sum of 5 occurs before a sum of 7 precisely if the events $E_n$ occurs for some $n$. Since $E_n$ and $E_m$ are disjoint if $n \neq m$, the desired probability is

$$
\sum_{n=1}^{\infty} P(E_n) = \sum_{n=1}^{\infty} \left( \frac{13}{18} \right)^{n-1} \cdot \frac{1}{9} \cdot \frac{1}{1 - \frac{13}{18}} = \frac{118}{9} \cdot \frac{5}{9} = \frac{2}{5}.
$$

Problem 27

$$
P(A \text{ wins in one move}) = \frac{3}{10}
$$

$$
P(A \text{ wins in three moves}) = \frac{7}{10} \cdot \frac{6}{9} \cdot \frac{3}{8} = \frac{7}{40}
$$

$$
P(A \text{ wins in five moves}) = \frac{7}{10} \cdot \frac{6}{9} \cdot \frac{5}{8} \cdot \frac{4}{7} \cdot \frac{3}{6} = \frac{1}{12}
$$

$$
P(A \text{ wins in seven moves}) = \frac{7}{10} \cdot \frac{6}{9} \cdot \frac{5}{8} \cdot \frac{4}{7} \cdot \frac{3}{6} \cdot \frac{2}{5} \cdot \frac{1}{4} = \frac{1}{40}
$$

$$
P(A \text{ wins}) = \frac{3}{10} + \frac{7}{40} + \frac{1}{12} + \frac{1}{40} = \frac{7}{12}
$$

Problem 28 (a) Without replacement:

$$
P(\text{all three balls are the same color}) = \frac{\binom{5}{3} + \binom{6}{3} + \binom{8}{3}}{\binom{19}{3}}
$$

With replacement:

$$
P(\text{all three balls are the same color}) = \left( \frac{5}{19} \right)^3 + \left( \frac{6}{19} \right)^3 + \left( \frac{8}{19} \right)^3
$$

(b) Without replacement:

$$
P(\text{all three balls are of different colors}) = \frac{\binom{5}{1} \cdot \binom{6}{1} \cdot \binom{8}{1}}{\binom{19}{3}}
$$

With replacement:

$$
P(\text{all three balls are of different colors}) = 3! \cdot \frac{5}{19} \cdot \frac{6}{19} \cdot \frac{8}{19}
$$

Problem 32 There are $(b+g)!$ ways to line up the children. There are $g \cdot (b+g-1)!$ arrangements with a girl in the $i$th position. The desired probability is

$$
\frac{g(b+g-1)!}{(b+g)!} = \frac{g}{b+g}.
$$
Problem 37  (a) There are \( \binom{10}{5} \) selections for the final exam. The number of selections that allow the student to solve all problems is \( \binom{7}{5} \), so that the desired probability is \( \frac{\binom{7}{5}}{\binom{10}{5}} = 0.08333 \).

(b) There are \( \binom{7}{4} \cdot \binom{3}{1} \) selections that’ll let the student solve exactly four problems, so that the probability of solving at least four problems is \( \frac{\binom{7}{4} \cdot \binom{3}{1}}{\binom{10}{5}} = \frac{1}{2} \).

Problem 43  (a) There are \( n! \) ways to arrange \( n \) people in a line. There are \( 2(n-1)! \) ways to arrange them such that \( A \) and \( B \) are next to each other. Hence, the probability of \( A \) and \( B \) being next to each other is \( \frac{2(n-1)!}{n!} = \frac{2}{n} \).

(b) If \( n = 2 \), then \( A \) and \( B \) will always be next to each other. Now, assume that \( n > 3 \). After \( A \) picks a seat, there are \( n-1 \) seats left, two of which are next to \( A \), so that the desired probability is \( \frac{2(n-1)!}{n!} \).

Problem 50 The probability that you have five spades and your partner has the remaining eight spades is

\[
\frac{\binom{13}{5} \cdot \binom{39}{8} \cdot \binom{8}{8} \cdot \binom{31}{5}}{\binom{52}{13} \cdot \binom{13}{2} \cdot \binom{26}{8}} = 2.6084 \cdot 10^{-6}.
\]

Problem 53 Let \( E_i \) be the event that the \( i \)-th couple sit together, for \( j = 1, \ldots, 4 \). Then \( P(E_i) = \frac{2}{8} = \frac{1}{4} \) (Problem 43(a)). Moreover, if \( i < j \), then \( P(E_i E_j) = \frac{2^4 \cdot 6!}{8!} \). Similarly, if \( i < j < k \), then \( P(E_i E_j E_k) = \frac{2^3 \cdot 5!}{8!} \).

Finally, we have \( P(E_1 E_2 E_3 E_4) = \frac{2^4 \cdot 4!}{8!} \). Using inclusion-exclusion, we obtain

\[
P(E_1 \cup E_2 \cup E_3 \cup E_4) = \sum_{i=1}^{4} P(E_i) - \sum_{i<j} P(E_iE_j) + \sum_{i<j<k} P(E_iE_jE_k) - P(E_1E_2E_3E_4)
\]

\[
= 4 \cdot \frac{1}{4} - \binom{4}{2} \frac{2^2 \cdot 6!}{8!} + \binom{4}{3} \frac{2^3 \cdot 5!}{8!} - \frac{2^4 \cdot 4!}{8!}
\]

\[
= 1 - \frac{3}{7} + \frac{2}{21} - \frac{1}{105} = \frac{23}{35}.
\]

Hence, the probability that no husband sits next to his wife is \( 1 - \frac{23}{35} = \frac{12}{35} \).
Problem 54 Let \( S, H, C, \) and \( D \) be the event that spades are missing, hearts are missing, etc. Then
\[
P(S \cup H \cup C \cup D) = P(S) + P(H) + P(C) + P(D)
\]
\[
- P(SH) - P(SC) - P(SD) - P(HC) - P(HD) - P(CD)
\]
\[
+ P(SHC) + P(SHD) + P(SCD) + P(HCD)
\]
\[
- P(SHCD)
\]
\[
= 4 \cdot \frac{39}{52} - 6 \cdot \frac{26}{52} + 4 \cdot \frac{1}{52} - 0
\]
\[
= 0.0511.
\]

Chapter 3

Problem 1 Let \( E \) be the event that at least one die lands on six, and let \( F \) be the event that the dice land of different numbers. Then
\[
P(EF) = 2 \cdot \frac{1}{6} \cdot \frac{5}{6} = \frac{5}{18}
\]
and
\[
P(F) = \frac{30}{36} = \frac{5}{6}.
\]
Hence,
\[
P(E|F) = \frac{P(EF)}{P(F)} = \frac{\frac{5}{18}}{\frac{5}{6}} = \frac{1}{3}.
\]

Problem 5
\[
\frac{6 \cdot 5 \cdot 9 \cdot 8}{15 \cdot 14 \cdot 13 \cdot 12} = \frac{6}{91}
\]

Problem 6 Let \( A \) be the event that the sample drawn contains exactly three white balls. Let \( B \) be the event that the first and third ball drawn are white.

Without replacement
\[
P(A) = 4 \cdot \frac{\frac{8}{12} \cdot \frac{7}{11} \cdot \frac{6}{10} \cdot \frac{4}{9}}{\frac{15}{12} \cdot \frac{14}{11} \cdot \frac{13}{10} \cdot \frac{12}{9}} \quad \text{and} \quad P(AB) = 2 \cdot \frac{\frac{8}{12} \cdot \frac{7}{11} \cdot \frac{6}{10} \cdot \frac{4}{9}}{\frac{15}{12} \cdot \frac{14}{11} \cdot \frac{13}{10} \cdot \frac{12}{9}},
\]
hence \( P(B|A) = \frac{P(AB)}{P(A)} = \frac{1}{2} \).

With replacement
\[
P(A) = \left(\frac{4}{3}\right) \left(\frac{2}{3}\right)^2 \frac{1}{3} \quad \text{and} \quad P(AB) = 2 \left(\frac{2}{3}\right)^2 \frac{1}{3},
\]
hence \( P(B|A) = \frac{P(AB)}{P(A)} = \frac{1}{2} \).

Problem 9 Let \( E_i \) be the event that the ball drawn from the \( i \)-th urn is white, for \( i = 1, 2, 3 \). Let \( F \) be the event that exactly two white balls were drawn.
Then
\[
P(E_1|F) = \frac{E_1F}{P(F)}
\]
\[
= \frac{P(E_1E_2E_3^c) + P(E_1E_3^cE_3)}{P(E_1E_2E_3^c) + P(E_1E_3^cE_3) + P(E_1^cE_2E_3)}
\]
\[
= \frac{\frac{2}{6}\cdot\frac{8}{12}\cdot\frac{3}{4} + \frac{2}{6}\cdot\frac{4}{12}\cdot\frac{1}{4}}{\frac{2}{6}\cdot\frac{8}{12}\cdot\frac{3}{4} + \frac{2}{6}\cdot\frac{4}{12}\cdot\frac{1}{4} + \frac{4}{6}\cdot\frac{8}{12}\cdot\frac{1}{4}}
\]
\[
= \frac{7}{11}.
\]

Problem 10 For \(i = 1, 2, 3\), let \(E_i\) be the event that the \(i\)-th card is a spade. Then

\[
P(E_1E_2E_3) = \frac{13}{52}\cdot\frac{12}{51}\cdot\frac{11}{50}
\]

and

\[
P(E_2E_3) = P(E_1E_2E_3) + P(E_1^cE_2E_3) = \frac{13}{52}\cdot\frac{12}{51}\cdot\frac{11}{50} + \frac{39}{52}\cdot\frac{13}{51}\cdot\frac{12}{50}.
\]

Thus

\[
P(E_1|E_2E_3) = \frac{P(E_1E_2E_3)}{P(E_2E_3)} = \frac{11}{50}.
\]