Math 461 Fall 2020

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General Info

8.2 Chebyshev’s inequality and the weak law of large numbers

8.3 Central Limit Theorem
Solution to Test 2 is on my homepage now. The median for Test 2 was 58.

HW10 is due Friday, 11/20, at noon. Please turn in your HW10 via the Moodle page. Make sure that your HW10 is uploaded successfully.
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Outline

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2. 8.2 Chebyshev’s inequality and the weak law of large numbers

3. 8.3 Central Limit Theorem
When we are given the distribution of a random variable $X$, we can find the probability of any event defined in terms of $X$. Suppose that we are only given the expectation and variance of $X$, then, in general, we can not find the probability of events defined in terms of $X$ exactly.

But for some events, we can still get some meaningful estimates on their probabilities. Let's first look at the case of a non-negative random variable $X$. Suppose that we only know $E[X]$. 
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But for some events, we can still get some meaningful estimates on their probabilities. Let’s first look at the case of a non-negative random variable $X$. Suppose that we only know $E[X]$. 
Markov inequality

Suppose that $X$ is a non-negative random variable, then for any $a > 0$,

$$P(X \geq a) \leq \frac{E[X]}{a}.$$  

Proof

Define a random variable

$$I = \begin{cases} 
1, & \text{if } X \geq a, \\
0, & \text{otherwise}.
\end{cases}$$

Then $I \leq X/a$. Thus

$$P(X \geq a) = E[I] \leq E[X/a] = \frac{E[X]}{a}.$$
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$$P(X \geq a) = E[I] \leq E[X/a] = \frac{E[X]}{a}.$$
Note that the Markov inequality gives a trivial bound when $a \leq E[X]$. It only gives a non-trivial bound for $a > E[X]$.

As a consequence of the Markov inequality, we have the following

### Chebyshev inequality

If $X$ is a random variable with finite mean $\mu$ and finite variance $\sigma^2$, then for any $\epsilon > 0$,

$$P(|X - \mu| \geq \epsilon) \leq \frac{\sigma^2}{\epsilon^2}.$$
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If $X$ is a random variable with finite mean $\mu$ and finite variance $\sigma^2$, then for any $\epsilon > 0$,

$$P(|X - \mu| \geq \epsilon) \leq \frac{\sigma^2}{\epsilon^2}.$$
Proof

Since \((X - \mu)^2\) is a non-negative random variable with mean \(\sigma^2\), we can apply the Markov inequality with \(a = \epsilon^2\) to get

\[
P(|X - \mu| \geq \epsilon) = P((X - \mu)^2 \geq \epsilon^2) \leq \frac{\sigma^2}{\epsilon^2}.
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Note that the Chebyshev inequality gives a trivial bound when \(\epsilon^2 \leq \sigma^2\). It only gives a non-trivial bound for \(\epsilon^2 > \sigma^2\).
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Example 1

Suppose that it is known that the number of items produced in a certain factory during a week is a random variable $X$ with mean $50$.

(i) What can be said about the probability that this week’s production will be at least $75$?

(ii) If the variance of a week’s production is known to be $25$, then what can be said about the probability that this week’s production will be between $40$ and $60$?

$$P(X \geq 75) \leq \frac{50}{75} = \frac{2}{3}.$$
Example 1

Suppose that it is known that the number of items produced in a certain factory during a week is a random variable $X$ with mean 50.

(i) What can be said about the probability that this week’s production will be at least 75?

(ii) If the variance of a week’s production is known to be 25, then what can be said about the probability that this week’s production will be between 40 and 60?

$$P(X \geq 75) \leq \frac{50}{75} = \frac{2}{3}.$$
Chebyshev's inequality, although very simple, is very useful. For example, it can be used to prove the following very important result, the weak law of large numbers.

$$P(40 \leq X \leq 60) = P(|x - 50| \leq 10) = 1 - P(|X - 50| \geq 11) \geq 1 - \frac{25}{11^2} = \frac{96}{121}.$$
Chebyshev’s inequality, although very simple, is very useful. For example, it can be used to prove the following very important result, the weak law of large numbers.

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Theorem (the weak law of large numbers)

Let $X_1, X_2, \ldots$ be a sequence of independent and identically distributed random variables with common (finite) mean $E[X_1] = \mu$. Then, for any $\epsilon > 0$,

$$P \left( \left| \frac{X_1 + \cdots + X_n}{n} - \mu \right| \geq \epsilon \right) \rightarrow 0 \quad \text{as} \quad n \rightarrow \infty.$$  

I will give a proof under the additional assumption that the random variables $X_1, X_2, \ldots$ have a finite variance $\sigma^2$. The proof in the general case is more difficult.
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Proof of the weak law of large numbers

Note that

\[ E \left( \frac{X_1 + \cdots + X_n}{n} \right) = \mu \]

and

\[ \text{Var} \left( \frac{X_1 + \cdots + X_n}{n} \right) = \frac{\sigma^2}{n}. \]

It follows from Chebyshev’s inequality that for any \( \epsilon > 0 \),

\[ P \left( \left| \frac{X_1 + \cdots + X_n}{n} - \mu \right| \geq \epsilon \right) \leq \frac{\sigma^2}{n\epsilon^2} \to 0, \quad \text{as } n \to \infty. \]
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3. 8.3 Central Limit Theorem
The central limit theorem is one the most important results in probability theory. In Chapter 5, we have already seen a special case of this result. Here is the general result

Central limit theorem
Suppose that $X_1, X_2, \ldots$ are independent and identically distributed random variables with common mean $\mu$ and common variance $\sigma^2$. Then the distribution of

$$\frac{X_1 + \cdots + X_n - n\mu}{\sigma \sqrt{n}}$$

tends to the standard normal distribution as $n \to \infty$. That is, for any $a \in \mathbb{R}$,

$$P\left( \frac{X_1 + \cdots + X_n - n\mu}{\sigma \sqrt{n}} \leq a \right) \to \Phi(a), \quad \text{as } n \to \infty.$$
The central limit theorem is one the most important results in probability theory. In Chapter 5, we have already seen a special case of this result. Here is the general result

**Central limit theorem**

Suppose that \( X_1, X_2, \ldots \) are independent and identically distributed random variables with common mean \( \mu \) and common variance \( \sigma^2 \). Then the distribution of

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P \left( \frac{X_1 + \cdots + X_n - n\mu}{\sigma \sqrt{n}} \leq a \right) \to \Phi(a), \quad \text{as } n \to \infty.
\]
Note that the generality of the theorem above. The common distributions of $X_1, X_2, \ldots$ can be discrete, can be continuous, and can be neither discrete nor continuous. It can be regarded as universality law. I will try to give a proof this result next time and give some applications.