Outline

1. General Info

2. 7.7 Moment generating functions
Solution to HW9 is is on my homepage now.

Test 2 is this Friday. Make sure that you make a reservation for Test 2 on CBTF. On Wed, I will do a brief review and then use answer questions.

Materials covered on Test 2: Section 4.9, Section 5.1, Section 5.2, Section 5.3, Section 5.4, Section 5.5, Section 5.6, Section 5.7, Section 6.1, Section 6.2, Section 6.3, Section 6.4, Section 6.5, Section 6.6, Section 7.2, Section 7.4
Solution to HW9 is is on my homepage now.

Test 2 is this Friday. Make sure that you make a reservation for Test 2 on CBTF. On Wed, I will do a brief review and then use answer questions.

Materials covered on Test 2: Section 4.9, Section 5.1, Section 5.2, Section 5.3, Section 5.4, Section 5.5, Section 5.6, Section 5.7, Section 6.1, Section 6.2, Section 6.3, Section 6.4, Section 6.5, Section 6.6, Section 7.2, Section 7.4
Solution to HW9 is on my homepage now.

Test 2 is this Friday. Make sure that you make a reservation for Test 2 on CBTF. On Wed, I will do a brief review and then use answer questions.

Materials covered on Test 2: Section 4.9, Section 5.1, Section 5.2, Section 5.3, Section 5.4, Section 5.5, Section 5.6, Section 5.7, Section 6.1, Section 6.2, Section 6.3, Section 6.4, Section 6.5, Section 6.6, Section 7.2, Section 7.4
Outline

1 General Info

2 7.7 Moment generating functions
The moment generating function of a random variable $X$ is defined to be the function

$$M_X(t) = E[e^{tx}], \quad t \in \mathbb{R}.$$ 

$M_X(t)$ may not be defined for all $t \in \mathbb{R}$, but it is always defined for $t = 0$. In fact, $M_X(0) = 1$. We will concentrate on random variables $X$ for which $M_X(t)$ is defined at least in an interval around the origin. All the important random variables we learned in the course satisfy this property.
The moment generating function of a random variable $X$ is defined to be the function

$$M_X(t) = E[e^{tx}], \quad t \in \mathbb{R}.$$ 

$M_X(t)$ may not be defined for all $t \in \mathbb{R}$, but it is always defined for $t = 0$. In fact, $M_X(0) = 1$. We will concentrate on random variables $X$ for which $M_X(t)$ is defined at least in an interval around the origin. All the important random variables we learned in the course satisfy this property.
Proposition

(i) If \( X \) is a binomial random variable with parameters \((n, p)\), then

\[ M_X(t) = (pe^t + 1 - p)^n, \quad \text{for all } t \in \mathbb{R}. \]

(ii) If \( X \) is a Poisson random variable with parameter \( \lambda \), then

\[ M_X(t) = e^{\lambda(e^t - 1)}, \quad \text{for all } t \in \mathbb{R}. \]

(iii) If \( X \) is a geometric random variable with parameter \( p \), then

\[ M_X(t) = \frac{pe^t}{1 - (1 - p)e^t}, \quad \text{for all } t < -\ln(1 - p). \]

(iv) If \( X \) is a negative binomial random variable with parameters \((r, p)\), then

\[ M_X(t) = \left(\frac{pe^t}{1 - (1 - p)e^t}\right)^r, \quad \text{for all } t < -\ln(1 - p). \]
Proposition (cont)

(v) If $X$ is uniformly distributed in the interval $(a, b)$, then

$$M_X(t) = \frac{e^{tb} - e^{ta}}{b - a}, \quad \text{for all } t \in \mathbb{R}.$$

(vi) If $X$ is an exponential random variable with parameter $\lambda$, then

$$M_X(t) = \frac{\lambda}{\lambda - t}, \quad \text{for all } t < \lambda.$$

(vii) If $X$ is a Gamma random variable with parameters $(\alpha, \lambda)$, then

$$M_X(t) = \left(\frac{\lambda}{\lambda - t}\right)^{\alpha}, \quad \text{for all } t < \lambda.$$

(viii) If $X$ is a normal random variable with parameters $(\mu, \sigma^2)$, then

$$M_X(t) = \exp \left(\mu t + \frac{\sigma^2 t^2}{2}\right), \quad \text{for all } t \in \mathbb{R}.$$
Theorem

If $X$ and $Y$ are two random variables with $M_X(t) = M_Y(t)$ for all $t$, then $X$ and $Y$ have the same distribution.

This theorem says that the moment generating function $M_X(t)$ of $X$ also contains all the statistical information about $X$.

Theorem

If $X$ and $Y$ are independent random variables, then

$$M_{X+Y}(t) = M_X(t)M_Y(t), \quad \text{for all } t.$$
Theorem

If \( X \) and \( Y \) are two random variables with \( M_X(t) = M_Y(t) \) for all \( t \), then \( X \) and \( Y \) have the same distribution.

This theorem says that the moment generating function \( M_X(t) \) of \( X \) also contains all the statistical information about \( X \).

Theorem

If \( X \) and \( Y \) are independent random variables, then

\[
M_{X+Y}(t) = M_X(t)M_Y(t), \quad \text{for all } t.
\]
Theorem
If \( X \) and \( Y \) are two random variables with \( M_X(t) = M_Y(t) \) for all \( t \), then \( X \) and \( Y \) have the same distribution.

This theorem says that the moment generating function \( M_X(t) \) of \( X \) also contains all the statistical information about \( X \).

Theorem
If \( X \) and \( Y \) are independent random variables, then

\[
M_{X+Y}(t) = M_X(t)M_Y(t), \quad \text{for all } t.
\]
Theorem

Suppose that $X$ and $Y$ are independent random variables.

(i) If $X$ is a binomial random variable with parameters $(m, p)$ and $Y$ is a binomial random variable with parameters $(n, p)$, then $X + Y$ is a binomial random variable with parameters $(m + n, p)$.

(ii) If $X$ is a Poisson random variable with parameter $\lambda_1$ and $Y$ is a Poisson random variable with parameter $\lambda_2$, then $X + Y$ is a Poisson random variable with parameters $\lambda_1 + \lambda_2$.

(iii) If $X$ is a negative binomial random variable with parameters $(r, p)$ and $Y$ is a negative binomial random variable with parameters $(s, p)$, then $X + Y$ is a negative binomial random variable with parameters $(r + s, p)$. 
### Theorem (cont)

**(iv)** If $X$ is a normal random variable with parameters $(\mu_1, \sigma_1^2)$ and $Y$ is a normal random variable with parameters $(\mu_2, \sigma_2^2)$, then $X + Y$ is a normal random variable with parameters $(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2)$.

**{(v)}** If $X$ is a Gamma random variable with parameters $(\alpha_1, \lambda)$ and $Y$ is a Gamma random variable with parameters $(\alpha_2, \lambda)$, then $X + Y$ is a Gamma random variable with parameters $(\alpha_1 + \alpha_2, \lambda)$.

The proofs of all five parts are the same. Let’s write out the details for the case of normal random variables.
Theorem (cont)

(iv) If $X$ is a normal random variable with parameters $(\mu_1, \sigma^2_1)$ and $Y$ is a normal random variable with parameters $(\mu_2, \sigma^2_2)$, then $X + Y$ is a normal random variable with parameters $(\mu_1 + \mu_2, \sigma^2_1 + \sigma^2_2)$.

(v) If $X$ is a Gamma random variable with parameters $(\alpha_1, \lambda)$ and $Y$ is a Gamma random variable with parameters $(\alpha_2, \lambda)$, then $X + Y$ is a Gamma random variable with parameters $(\alpha_1 + \alpha_2, \lambda)$.

The proofs of all five parts are the same. Let’s write out the details for the case of normal random variables.
We know that

\[ M_X(t) = \exp \left( \mu_1 t + \frac{\sigma_1^2 t^2}{2} \right), \quad M_Y(t) = \exp \left( \mu_2 t + \frac{\sigma_2^2 t^2}{2} \right). \]

Since \( X \) and \( Y \) are independent,

\[
M_{X+Y}(t) = M_X(t)M_Y(t)
\]

\[
= \exp \left( \mu_1 t + \frac{\sigma_1^2 t^2}{2} \right) \exp \left( \mu_2 t + \frac{\sigma_2^2 t^2}{2} \right)
\]

\[
= \exp \left( (\mu_1 + \mu_2)t + \frac{(\sigma_1^2 + \sigma_2^2)t^2}{2} \right).
\]

Hence \( X + Y \) is a normal random variable with parameters \( (\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2) \).
Example

Suppose that $X_1, X_2, \ldots$ are independent Bernoulli random variables with parameter $p \in (0, 1)$, and that $N$ is a Poisson random variable with parameter $\lambda$ independent of all the $X_i$'s. Define

$$Y = \sum_{i=1}^{N} X_i$$

with the convention $\sum_{i=1}^{0} = 0$. Find $E[Y]$ and $\text{Var}(Y)$.

Let try to find out the moment generating function of $Y$. 
Example

Suppose that $X_1, X_2, \ldots$ are independent Bernoulli random variables with parameter $p \in (0, 1)$, and that $N$ is a Poisson random variable with parameter $\lambda$ independent of all the $X_i$’s. Define

$$Y = \sum_{i=1}^{N} X_i$$

with the convention $\sum_{i=1}^{0} = 0$. Find $E[Y]$ and $\text{Var}(Y)$.

Let try to find out the moment generating function of $Y$. 
\[ M_Y(t) = E \left[ \exp \left( t \sum_{i=1}^{N} X_i \right) \right] = E \left[ E \left[ \exp \left( t \sum_{i=1}^{N} X_i \right) \mid N \right] \right] \]

\[ = \sum_{n=0}^{\infty} E \left[ \exp \left( t \sum_{i=1}^{n} X_i \right) \mid N = n \right] P(N = n) \]

\[ = \sum_{n=0}^{\infty} E \left[ \exp \left( t \sum_{i=1}^{n} X_i \right) \right] P(N = n) \]

\[ = \sum_{n=0}^{\infty} \left( p e^t + 1 - p \right)^n e^{-\lambda} \frac{\lambda^n}{n!} \]

\[ = e^{\lambda p (e^t - 1)} \]
Thus $Y$ is a Poisson random variable with parameter $\lambda p$. Hence

$$E[Y] = \text{Var}(Y) = \lambda p.$$
Thus $Y$ is a Poisson random variable with parameter $\lambda p$. Hence

$$E[Y] = \text{Var}(Y) = \lambda p.$$
Let $M_X(t)$ be the common moment generating function of $X_1, X_2, \ldots$.

$$M_Y(t) = E \left[ \exp \left( t \sum_{i=1}^{N} X_i \right) \right] = E \left[ \exp \left( t \sum_{i=1}^{N} X_i \right) | N \right]$$

$$= \sum_{n=0}^{\infty} E \left[ \exp \left( t \sum_{i=1}^{n} X_i \right) | N = n \right] P(N = n)$$

$$= \sum_{n=0}^{\infty} \left[ \exp \left( t \sum_{i=1}^{n} X_i \right) \right] P(N = n)$$

$$= \sum_{n=0}^{\infty} (M_X(t))^n P(N = n)$$

$$= E[(M_X(t))^N].$$
Taking derivative, we get

\[ M'_Y(t) = E[N(M_X(t))^{N-1} M'_X(t)]. \]

Hence

\[ M'_Y(0) = E[N(M_X(0))^{N-1} M'_X(0)] = E[N E[X]] = E[N] E[X]. \]

Taking derivative again, we get

\[ M''_Y(t) = E[N(N-1)(M_X(t))^{N-2}(M'_X(t))^2 + N(M_X(t))^{N-1} M''_X(t)]. \]
Taking derivative, we get

\[ M'_Y(t) = E[N (M_X(t))^{N-1} M'_X(t)]. \]

Hence

\[ M'_Y(0) = E[N (M_X(0))^{N-1} M'_X(0)] = E[NE[X]] = E[N]E[X]. \]

Taking derivative again, we get

\[ M''_Y(t) = E[N(N - 1) (M_X(t))^{N-2} (M'_X(t))^2 + N (M_X(t))^{N-1} M''_X(t)]. \]
Thus

\[ E[Y^2] = M''_Y(0) = E[N(N - 1)(E[X])^2 + NE[X^2]] \]
\[ = (E[X])^2 (E[N^2] - E[N]) + E[N]E[X^2] \]
\[ = E[N](E[X^2] - (E[X])^2) + (E[X])^2 E[N^2] \]
\[ = E[N]\text{Var}(X) + (E[X])^2 E[N^2]. \]

Therefore

\[ \text{Var}(Y) = E[N]\text{Var}(X) + (E[X])^2 E[N^2] - (E[X])^2 (E[N])^2 \]
\[ = E[N]\text{Var}(X) + (E[X])^2 \text{Var}(N). \]
Thus

\[ E[Y^2] = M''_Y(0) = E[N(N - 1)(E[X])^2 + NE[X^2]] \]
\[ = (E[X])^2 (E[N^2] - E[N]) + E[N]E[X^2] \]
\[ = E[N](E[X^2] - (E[X])^2) + (E[X])^2 E[N^2] \]
\[ = E[N] \text{Var}(X) + (E[X])^2 E[N^2]. \]

Therefore

\[ \text{Var}(Y) = E[N] \text{Var}(X) + (E[X])^2 E[N^2] - (E[X])^2 (E[N])^2 \]
\[ = E[N] \text{Var}(X) + (E[X])^2 \text{Var}(N). \]