HW9 is due Friday, 11/06, at noon. Please submit your HW via the course Moodle page. Make sure that your HW is uploaded successfully.

Solution to HW8 is on my homepage.
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Outline

1 General Info

2 7.4 Covariance, variance of sums and correlations
Now we are going to use the formula

\[ \text{Var}(\sum_{i=1}^{n} X_i) = \sum_{i=1}^{n} \text{Var}(X_i) + \sum_{i \neq j} \text{Cov}(X_i, X_j) \]

\[ = \sum_{i=1}^{n} \text{Var}(X_i) + 2 \sum_{i<j} \text{Cov}(X_i, X_j). \]

to find the variance of some complicated random variables.

**Example 8**

A group of \( N \) people throw their hats into the center of the room. The hats are mixed up, and each person randomly selects a hat. Let \( X \) be the number of people who get their own hats. Find \( \text{Var}(X) \).
Now we are going to use the formula

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$$= \sum_{i=1}^{n} \text{Var}(X_i) + 2 \sum_{i < j} \text{Cov}(X_i, X_j).$$

to find the variance of some complicated random variables.

**Example 8**

A group of $N$ people throw their hats into the center of the room. The hats are mixed up, and each person randomly selects a hat. Let $X$ be the number of people who get their own hats. Find $\text{Var}(X)$. 
For \( i = 1, \ldots, N \), let \( X_i = 1 \) if the \( i \)-th man gets his own hat and \( X_i = 0 \) otherwise. Then \( X = X_1 + \cdots + X_N \). Note that

\[
E[X_i] = P(X_i = 1) = \frac{1}{N}, \quad \text{Var}(X_i) = \frac{1}{N} \left(1 - \frac{1}{N}\right).
\]

Now let’s find \( \text{Cov}(X_i, X_j) \) for \( i \neq j \). \( X_iX_j \) is also a Bernoulli random variable.

\[
E[X_iX_j] = P(X_iX_j = 1) = P(X_i = 1, X_j = 1) = \frac{1}{N(N-1)}.
\]

Thus

\[
\text{Cov}(X_i, X_j) = \frac{1}{N(N-1)} - \frac{1}{N^2}.
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Var(\(X\)) = Var(\(\sum_{i=1}^{N} X_i\))

\[= \sum_{i=1}^{N} \text{Var}(X_i) + \sum_{i \neq j} \text{Cov}(X_i, X_j)\]

\[= N \cdot \frac{1}{N} \left(1 - \frac{1}{N}\right) + N(N - 1) \left(\frac{1}{N(N - 1)} - \frac{1}{N^2}\right)\]

\[= 1.\]

**Example 9**

If \(n\) balls are randomly selected, without replacement, from a box containing \(N\) (\(N > n\)) balls, of which \(m\) are white, find the expected number of white balls selected.
Var(\(X\)) = Var(\(\sum_{i=1}^{N} X_i\))

= \sum_{i=1}^{N} Var(X_i) + \sum_{i \neq j} Cov(X_i, X_j)

= \frac{1}{N} \left(1 - \frac{1}{N}\right) + \frac{1}{N-1} \left(\frac{1}{N(N-1)} - \frac{1}{N^2}\right)

= 1.

**Example 9**

If \(n\) balls are randomly selected, without replacement, from a box containing \(N\) (\(N > n\)) balls, of which \(m\) are white, find the expected number of white balls selected.
For $i = 1, \ldots, m$, let $Y_i = 1$ if the $i$-th white ball is among the selected. Then $X = Y_1 + \cdots + Y_m$. Note that

$$E[Y_i] = \frac{(\frac{N-1}{n-1})}{\binom{N}{n}} = \frac{n}{N}, \quad \text{Var}(Y_i) = \frac{n}{N} \left(1 - \frac{n}{N}\right).$$

Now let’s find $\text{Cov}(Y_i, Y_j)$ for $i \neq j$. $Y_i Y_j$ is also a Bernoulli random variable.

$$E[Y_i Y_j] = P(Y_i = 1, Y_j = 1) = \frac{(\frac{N-2}{n-2})}{\binom{N}{n}} = \frac{n(n-1)}{N(N-1)}.$$

Thus

$$\text{Cov}(Y_i, Y_j) = \frac{n(n-1)}{N(N-1)} - \frac{n^2}{N^2}.$$
For \( i = 1, \ldots, m \), let \( Y_i = 1 \) if the \( i \)-th white ball is among the selected. Then \( X = Y_1 + \cdots + Y_m \). Note that

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\]

Thus

\[
\text{Cov}(Y_i, Y_j) = \frac{n(n - 1)}{N(N - 1)} - \frac{n^2}{N^2}.
\]
\[ \text{Var}(X) = \text{Var}(\sum_{i=1}^{m} Y_i) \]

\[ = \sum_{i=1}^{m} \text{Var}(Y_i) + \sum_{i \neq j} \text{Cov}(Y_i, Y_j) \]

\[ = m \frac{n}{N} \left( 1 - \frac{n}{N} \right) + m(m - 1) \left( \frac{n(n - 1)}{N(N - 1)} - \frac{n^2}{N^2} \right). \]

**Example 10**

\( n \) balls are randomly distributed into \( r \) boxes (so that each ball is equally likely to go to any of the \( r \) boxes). Let \( X \) be the number of empty boxes. Find \( \text{Var}(X) \).
\[
\text{Var}(X) = \text{Var}\left( \sum_{i=1}^{m} Y_i \right)
\]
\[
= \sum_{i=1}^{m} \text{Var}(Y_i) + \sum_{i \neq j} \text{Cov}(Y_i, Y_j)
\]
\[
= mn \left( 1 - \frac{n}{N} \right) + m(m - 1) \left( \frac{n(n - 1)}{N(N - 1)} - \frac{n^2}{N^2} \right).
\]

**Example 10**

\(n\) balls are randomly distributed into \(r\) boxes (so that each ball is equally likely to go to any of the \(r\) boxes). Let \(X\) be the number of empty boxes. Find \(\text{Var}(X)\).
For $i = 1, \ldots, r$, let $X_i = 1$ if box number $i$ is empty and $X_i = 0$ otherwise. Then $X = X_1 + \cdots + X_r$. Note that for $i = 1, \ldots, r$,

$$E[X_i] = P(X_i = 1) = \left( \frac{r-1}{r} \right)^n, \quad \text{Var}(X_i) = \left( \frac{r-1}{r} \right)^n \left( 1 - \left( \frac{r-1}{r} \right)^n \right).$$

Now let’s find $\text{Cov}(X_i, X_j)$ for $i \neq j$. $X_iX_j$ is also a Bernoulli random variable.

$$E[X_iX_j] = P(X_i = 1, X_j = 1) = \left( \frac{r-2}{r} \right)^n.$$

Thus

$$\text{Cov}(X_i, X_j) = \left( \frac{r-2}{r} \right)^n - \left( \frac{r-1}{r} \right)^{2n}.$$
For $i = 1, \ldots, r$, let $X_i = 1$ if box number $i$ is empty and $X_i = 0$ otherwise. Then $X = X_1 + \cdots + X_r$. Note that for $i = 1, \ldots, r$,

$$E[X_i] = P(X_i = 1) = \left(\frac{r - 1}{r}\right)^n, \quad \text{Var}(X_i) = \left(\frac{r - 1}{r}\right)^n \left(1 - \left(\frac{r - 1}{r}\right)^n\right).$$

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$$\text{Cov}(X_i, X_j) = \left(\frac{r - 2}{r}\right)^n - \left(\frac{r - 1}{r}\right)^{2n}.$$
\[ \text{Var}(X) = \text{Var}(\sum_{i=1}^{r} X_i) \]
\[ = \sum_{i=1}^{r} \text{Var}(X_i) + \sum_{i \neq j} \text{Cov}(X_i, X_j) \]
\[ = r \left( \frac{r - 1}{r} \right)^n \left( 1 - \left( \frac{r - 1}{r} \right)^n \right) + r(r - 1) \left( \left( \frac{r - 2}{r} \right)^n - \left( \frac{r - 1}{r} \right)^{2n} \right). \]

**Example 11**

There are \( n \) types of coupons. Each newly obtained coupon is, independently, equally like to be any of the \( n \) types. Let \( X \) be the number of distinct types contained in a collection of \( k \) coupons. Find \( \text{Var}(X) \).
\[ \text{Var}(X) = \text{Var} \left( \sum_{i=1}^{r} X_i \right) \]

\[ = \sum_{i=1}^{r} \text{Var}(X_i) + \sum_{i \neq j} \text{Cov}(X_i, X_j) \]

\[ = r \left( \frac{r - 1}{r} \right)^n \left( 1 - \left( \frac{r - 1}{r} \right)^n \right) + r(r - 1) \left( \left( \frac{r - 2}{r} \right)^n - \left( \frac{r - 1}{r} \right)^{2n} \right). \]

**Example 11**

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For $i = 1, \cdots, n$, let $X_i = 1$ if there is at least one type $i$ coupon in the collection of $k$ coupons and $X_i = 0$ otherwise. Then $X_1 + \cdots + X_n$ is the number of distinct types in the collection of $k$ coupons.

For $i = 1, \cdots, n,$

$$P(X_i = 0) = \left(1 - \frac{1}{n}\right)^k, \quad P(X_i = 1) = 1 - \left(1 - \frac{1}{n}\right)^k.$$ 

For $i \neq j$,

$$P(X_iX_j = 0) = P(X_i = 0) + P(X_j = 0) - P(X_i = 0, X_j = 0) = 2 \left(1 - \frac{1}{n}\right)^k - \left(1 - \frac{2}{n}\right)^k,$$

$$P(X_iX_j = 1) = 1 - 2 \left(1 - \frac{1}{n}\right)^k + \left(1 - \frac{2}{n}\right)^k.$$
For \( i = 1, \cdots, n \), let \( X_i = 1 \) if there is at least one type \( i \) coupon in the collection of \( k \) coupons and \( X_i = 0 \) otherwise. Then \( X_1 + \cdots + X_n \) is the number of distinct types in the collection of \( k \) coupons.

For \( i = 1, \cdots, n \),

\[
P(X_i = 0) = \left(1 - \frac{1}{n}\right)^k, \quad P(X_i = 1) = 1 - \left(1 - \frac{1}{n}\right)^k.
\]

For \( i \neq j \),

\[
P(X_iX_j = 0) = P(X_i = 0) + P(X_j = 0) - P(X_i = 0, X_j = 0)
\]

\[
= 2\left(1 - \frac{1}{n}\right)^k - \left(1 - \frac{2}{n}\right)^k,
\]

\[
P(X_iX_j = 1) = 1 - 2\left(1 - \frac{1}{n}\right)^k + \left(1 - \frac{2}{n}\right)^k.
\]
Thus

\[ E[X_i] = 1 - \left(1 - \frac{1}{n}\right)^k, \quad \text{Var}(X_i) = \left(1 - \frac{1}{n}\right)^k \left(1 - \left(1 - \frac{1}{n}\right)^k\right), \]

and

\[ \text{Cov}(X_i, X_j) = 1 - 2 \left(1 - \frac{1}{n}\right)^k + \left(1 - \frac{2}{n}\right)^k - \left(1 - \left(1 - \frac{1}{n}\right)^k\right)^2. \]
Consequently

\[ E[X_1 + \cdots + X_n] = n \left( 1 - \left( 1 - \frac{1}{n} \right)^k \right) \]

and

\[ \text{Var}(X_1 + \cdots + X_n) = n \left( 1 - \frac{1}{n} \right)^k \left( 1 - \left( 1 - \frac{1}{n} \right)^k \right) \]

\[ + n(n-1) \left( 1 - 2 \left( 1 - \frac{1}{n} \right)^k + \left( 1 - \frac{2}{n} \right)^k - \left( 1 - \left( 1 - \frac{1}{n} \right)^k \right)^2 \right) . \]
Example 12

10 couples are randomly seated at a round table. Let $X$ be the number of couples that are seated together. Find $\text{Var}(X)$.

For $i = 1, \ldots, 10$, let $X_i = 1$ if the $i$-th couple are seated together. Then $X = X_1 + \cdots + X_{10}$. For $i = 1, \ldots, 10$,

$$P(X_i = 1) = \frac{2(18)!}{(19)!} = \frac{2}{19}.$$ 

For $i \neq j$,

$$P(X_i = 1, X_j = 1) = \frac{2^2(17)!}{(19)!} = \frac{4}{19 \cdot 18}.$$
Example 12

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For $i = 1, \ldots, 10$, let $X_i = 1$ if the $i$-the couple are seated together. Then $X = X_1 + \cdots + X_{10}$. For $i = 1, \ldots, 10,$

$$P(X_i = 1) = \frac{2(18)!}{(19)!} = \frac{2}{19}.$$ 

For $i \neq j$,

$$P(X_i = 1, X_j = 1) = \frac{2^2(17)!}{(19)!} = \frac{4}{19 \cdot 18}. $$
Thus for $i = 1, \ldots, 10$, 

$$E[X_i] = \frac{2}{19}, \quad \text{Var}(X_i) = \frac{2}{19} \frac{17}{19}.$$ 

For $i \neq j$, 

$$\text{Cov}(X_i, X_j) = \frac{4}{19 \cdot 18} - \frac{4}{19^2}.$$ 

$$\text{Var}(X) = \text{Var} \left( \sum_{i=1}^{10} X_i \right)$$ 

$$= \sum_{i=1}^{10} \text{Var}(X_1) + \sum_{i \neq j} \text{Cov}(X_i, X_j)$$ 

$$= \frac{20}{19} \frac{17}{19} + 90 \left( \frac{4}{19 \cdot 18} - \frac{4}{19^2} \right).$$
Thus for \( i = 1, \ldots, 10, \)

\[
E[X_i] = \frac{2}{19}, \quad \text{Var}(X_i) = \frac{2}{19} \frac{17}{19}.
\]

For \( i \neq j, \)

\[
\text{Cov}(X_i, X_j) = \frac{4}{19 \cdot 18} - \frac{4}{19^2}.
\]

\[
\text{Var}(X) = \text{Var}(\sum_{i=1}^{10} X_i)
= \sum_{i=1}^{10} \text{Var}(X_1) + \sum_{i \neq j} \text{Cov}(X_i, X_j)
= \frac{20}{19} \frac{17}{19} + 90 \left( \frac{4}{19 \cdot 18} - \frac{4}{19^2} \right).
\]