Math 461 Fall 2020

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October 23, 2020
HW7 is due today, at noon. Please submit your HW via the course Moodle page. Make sure that your HW is uploaded successfully.

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Outline

1. General Info

2. 6.3 Sums of independent random variables
Last time, we have seen that, if $X$ and $Y$ are independent abs. cont. random variables with density $f_X$ and $f_Y$ respectively, then the density of $Z = X + Y$ is

$$f_Z(z) = \int_{-\infty}^{\infty} f_X(x)f_Y(z - x)dx$$

We also have

$$f_Z(z) = \int_{-\infty}^{\infty} f_X(z - y)f_Y(y)dy.$$

Now let’s suppose that $X$ and $Y$ are independent positive abs. cont. random variables with density $f_X$ and $f_Y$ respectively, then $Z = X + Y$ is a also a positive random variable and its density is
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\[ f_Z(z) = \begin{cases} \int_0^z f_X(x)f_Y(z-x)\,dx, & z > 0, \\ 0, & \text{otherwise.} \end{cases} \]

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**Proposition**

Suppose \( X \) and \( Y \) are independent random variables.

(i) If \( X \) and \( Y \) are Gamma random variables with parameters \((\alpha, \lambda)\) and \((\beta, \lambda)\) respectively, then \( X + Y \) is a Gamma random variable with parameters \((\alpha + \beta, \lambda)\).

(ii) If \( X \) and \( Y \) are normal random variables with parameters \((\mu_1, \sigma_1^2)\) and \((\mu_2, \sigma_2^2)\) respectively, then \( X + Y \) is a normal random variable \((\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2)\).
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Let’s prove (i). For any $z > 0$,

$$f_{X+Y}(z) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_0^z \lambda e^{-\lambda x} (\lambda x)^{\alpha-1} \lambda e^{-\lambda(z-x)} (\lambda(z-x))^{\beta-1} dx$$

$$= \frac{\lambda e^{-\lambda z}}{\Gamma(\alpha)\Gamma(\beta)} \lambda^{\alpha+\beta-1} \int_0^z x^{\alpha-1} (z-x)^{\beta-1} dx$$

$$= \frac{\lambda e^{-\lambda z}}{\Gamma(\alpha)\Gamma(\beta)} (\lambda z)^{\alpha+\beta-1} \int_0^1 u^{\alpha-1} (1-u)^{\beta-1} du, \quad x = zu,$$

$$= \frac{\lambda e^{-\lambda z}}{\Gamma(\alpha)\Gamma(\beta)} (\lambda z)^{\alpha+\beta-1} B(\alpha, \beta)$$

$$= \frac{1}{\Gamma(\alpha + \beta)} \lambda e^{-\lambda z} (\lambda z)^{\alpha+\beta-1}.$$
Example 1

A basketball team will play a 44-game season. 26 of these games are against class A teams and 18 are against class B teams. Suppose that the team will win each game against a class A team with probability .4 and will win each game against a class B team with probability .7. Suppose also that the results of different games are independent. Approximate the probability that

(a) the team wins 25 or more games;
(b) the team will win more games against class A teams than it does against class B teams.

Let $X_A$ and $X_B$ denote respectively the number of games the team wins are against class A teams and are against class B teams. Then $X_A$ and $X_B$ are independent binomial random variables with parameters (26, .4) and (18, .7) respectively.
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\begin{align*}
E[X_A] &= 26(.4) = 10.4, \quad \text{Var}(X_A) = 26(.4)(.6) = 6.24 \\
E[X_B] &= 18(.7) = 12.6, \quad \text{Var}(X_B) = 18(.7)(.3) = 3.78.
\end{align*}

By the central limit theorem, \( X_A \) is approximately normal with parameters \((10.4, 6.24)\) and \( X_B \) is approximately normal with parameters \((12.6, 3.78)\).

By the Proposition above, \( X_A + X_B \) is approximately normal with parameters \((23, 10.02)\) since \( X_A \) and \( X_B \) are independent. Thus

\begin{align*}
P(X_A + X_B \geq 25) &= P(X_A + X_B \geq 24.5) \\
&= P \left( \frac{X_A + X_B - 23}{\sqrt{10.02}} \geq \frac{24.5 - 23}{\sqrt{10.02}} \right) \\
&= P \left( \frac{X_A + X_B - 23}{\sqrt{10.02}} \geq .4739 \right) \approx 1 - \Phi(.4739) \approx .3178.
\end{align*}
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= P \left( \frac{X_A + X_B - 23}{\sqrt{10.02}} \geq .4739 \right) \approx 1 - \Phi(.4739) \approx .3178.
\]
Since $X_A$ and $X_B$ are independent, by the Proposition above, $X_A - X_B$ is approximately normal with parameters $(-2.2, 10.02)$. Hence

$$P(X_A - X_B \geq 1) = P(X_A - X_B \geq .5)$$

$$= P \left( \frac{X_A - X_B + 2.2}{\sqrt{10.02}} \geq \frac{.5 + 2.2}{\sqrt{10.02}} \right)$$

$$= P \left( \frac{X_A - X_B + 2.2}{\sqrt{10.02}} \geq .8530 \right) \approx 1 - \Phi(.8530) \approx .1968.$$

**Example 2**

Suppose that $X$ and $Y$ are independent standard normal random variables. Find the density of $Z = X^2 + Y^2$. 
Since $X_A$ and $X_B$ are independent, by the Proposition above, $X_A - X_B$ is approximately normal with parameters $(-2.2, 10.02)$. Hence

\[
P(X_A - X_B \geq 1) = P(X_A - X_B \geq 0.5)
= P \left( \frac{X_A - X_B + 2.2}{\sqrt{10.02}} \geq \frac{0.5 + 2.2}{\sqrt{10.02}} \right)
= P \left( \frac{X_A - X_B + 2.2}{\sqrt{10.02}} \geq 0.8530 \right) \approx 1 - \Phi(0.8530) \approx 0.1968.
\]

**Example 2**

Suppose that $X$ and $Y$ are independent standard normal random variables. Find the density of $Z = X^2 + Y^2$. 
We know that $X^2$ and $Y^2$ are independent Gamma random variables with parameters $(\frac{1}{2}, \frac{1}{2})$. Thus $X^2 + Y^2$ is a Gamma random variables with parameters $(1, \frac{1}{2})$, that is, an exponential random variable with parameter $1/2$.

**Example 3**

Suppose that $X$ and $Y$ are independent random variables, both uniformly distributed on $(0, 1)$. Find the density of $Z = X + Y$.

Applying the formula directly is not easy. We look for the distribution of $Z$ first.
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Applying the formula directly is not easy. We look for the distribution of $Z$ first.
$X + Y$ takes values in $(0, 2)$. For $z \in (0, 1)$,

$$P(Z \leq z) = P(X + Y \leq z) = \frac{z^2}{2}.$$ 

For $z \in (1, 2)$,

$$P(Z \leq z) = P(X + Y \leq z) = 1 - \frac{(2 - z)^2}{2}.$$
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For $z \in (1, 2)$,

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Thus the density of $Z$ is

$$f_Z(z) = \begin{cases} 
  z, & 0 \leq z \leq 1, \\
  2 - z, & 1 < z < 2, \\
  0, & \text{otherwise.}
\end{cases}$$

Suppose that $X$ and $Y$ are independent discrete random variables with mass functions $p_X(\cdot)$ and $p_Y(\cdot)$ respectively. Find the mass function of $Z = X + Y$. 
Thus the density of $Z$ is

$$f_Z(z) = \begin{cases} 
  z, & 0 \leq z \leq 1, \\
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\end{cases}$$

Suppose that $X$ and $Y$ are independent discrete random variables with mass functions $p_X(\cdot)$ and $p_Y(\cdot)$ respectively. Find the mass function of $Z = X + Y$. 
For any $z$,

$$p_Z(z) = P(X + Y = z) = \sum_x P(X + Y = z, X = x)$$

$$= \sum_x P(X = x, Y = z - x) = \sum_x P(X = x)P(Y = z - x)$$

$$= \sum_x p_X(x)p_Y(z - x).$$

We also have

$$p_Z(z) = \sum_y p_X(z - y)p_Y(y).$$
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We also have

$$p_Z(z) = \sum_y p_X(z - y)p_Y(y).$$
If $X$ and $Y$ are integer-valued, then for any integer $z$,

$$p_{X+Y}(z) = \sum_{x=-\infty}^{\infty} p_X(x)p_Y(z-x).$$

If $X$ and $Y$ are non-negative integer-valued, then for any non-negative integer $z$,

$$p_{X+Y}(z) = \sum_{x=0}^{z} p_X(x)p_Y(z-x).$$

If $X$ and $Y$ are positive integer-valued, then $X + Y$ takes values $2, 3, \ldots$. For $z = 2, 3, \ldots$,

$$p_{X+Y}(z) = \sum_{x=1}^{z-1} p_X(x)p_Y(z-x).$$
If $X$ and $Y$ are integer-valued, then for any integer $z$,

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If $X$ and $Y$ are positive integer-valued, then $X+Y$ takes values $2, 3, \ldots$. For $z = 2, 3, \ldots$,

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Proposition

Suppose that $X$ and $Y$ are independent random variables.

(i) If $X$ is a binomial random variable with parameters $(m, p)$, and $Y$ is a binomial random variable with parameters $(n, p)$, then $X + Y$ is a binomial random variable with parameters $(m + n, p)$;

(ii) If $X$ is a Poisson random variables with parameter $\lambda_1$, and $Y$ is a Poisson random variables with parameter $\lambda_2$, then $X + Y$ is a Poisson random variables with parameter $\lambda_1 + \lambda_2$;

(iii) If $X$ is a negative binomial random variable with parameters $(r_1, p)$, and $Y$ is a negative binomial random variable with parameters $(r_2, p)$, then $X + Y$ is a negative binomial random variable with parameters $(r_1 + r_2, p)$.

I will only give the proof of (ii).
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I will only give the proof of (ii).
For any \( z = 0, 1, \ldots \),

\[
p_{X+Y}(z) = \sum_{x=0}^{z} e^{-\lambda_1} \frac{\lambda_1^x}{x!} e^{-\lambda_2} \frac{\lambda_2^{z-x}}{(z-x)!}
\]

\[
= e^{-(\lambda_1+\lambda_2)} \frac{(\lambda_1 + \lambda_2)^z}{z!} \sum_{x=0}^{z} \binom{z}{x} \left( \frac{\lambda_1}{\lambda_1 + \lambda_2} \right)^x \left( \frac{\lambda_2}{\lambda_1 + \lambda_2} \right)^{z-x}
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**Example 4**

Suppose that \( X \) and \( Y \) are independent geometric random variables with a common parameter \( p \). Find (a) the mass function of \( \min(X, Y) \); (b) \( P(\min(X, Y) = X) = P(Y \geq X) \).
For any \( z = 0, 1, \ldots \),

\[
px+y(z) = \sum_{x=0}^{z} e^{-\lambda_1} \frac{\lambda_1^x}{x!} e^{-\lambda_2} \frac{\lambda_2^{z-x}}{(z-x)!} \\
= e^{-(\lambda_1+\lambda_2)} \frac{(\lambda_1 + \lambda_2)^z}{z!} \sum_{x=0}^{z} \binom{z}{x} \left( \frac{\lambda_1}{\lambda_1 + \lambda_2} \right)^x \left( \frac{\lambda_2}{\lambda_1 + \lambda_2} \right)^{z-x} \\
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**Example 4**

Suppose that \( X \) and \( Y \) are independent geometric random variables with a common parameter \( p \). Find (a) the mass function of \( \min(X, Y) \); (b) \( P(\min(X, Y) = X) = P(Y \geq X) \).
min\( (X, Y) \) takes only positive integer values. For \( z = 1, 2, \ldots \),

\[
P(\min(X, Y) > z) = P(X > z, Y > z) = P(X > z)P(Y > z)
= (1 - p)^{2z} = (1 - (2p - p^2))^z.
\]

Thus \( \min(X, Y) \) is a geometric random variable with parameter \( 2p - p^2 \).

\[
P(Y \geq X) = \sum_{x=1}^{\infty} P(X = x, Y \geq x) = \sum_{x=1}^{\infty} P(X = x, Y \geq x)
= \sum_{x=1}^{\infty} P(X = x)P(Y \geq x) = \sum_{x=1}^{\infty} p(1 - p)^{x-1}(1 - p)^{x-1}
= p \sum_{x=1}^{\infty} (1 - (2p - p^2))^{x-1} = \frac{p}{2p - p^2} = \frac{1}{2 - p}.
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min(X, Y) takes only positive integer values. For \( z = 1, 2, \ldots \),

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\]
Suppose that $X$ and $Y$ are independent random variables such that

$$P(X = i) = P(Y = i) = \frac{1}{100}, \quad i = 1, \ldots, 100.$$ 

Find (a) $P(X \geq Y)$; (b) $P(X = Y)$.

\[ P(X \geq Y) = \sum_{y=1}^{100} P(X \geq Y, Y = y) = \sum_{y=1}^{100} P(X \geq y) P(Y = y) \]

\[ = \frac{1}{100^2} \sum_{y=1}^{100} (101 - y) = \frac{1}{100^2} \sum_{i=1}^{100} i = \frac{101}{200}. \]

\[ P(X = Y) = \sum_{y=1}^{100} P(X = x, Y = x) = \sum_{y=1}^{100} P(X = x, Y = x) \]

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